MUMBAI UNIVERSITY

SEMESTER 2

APPLIED MATHEMATICS SOLVED PAPER – MAY 2017

N.B:- (1) Question no. 1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1.(a) **Evaluate** $\int_0^\infty 3^{-4x^2}\ dx$ **[3]**

Ans: **Let** I = $\int_0^\infty 3^{-4x^2}\ dx$

put $3^{-4x^2} = e^{-t}$

taking log on both sides,

$$4x^2 \log 3 = t$$

$$x^2 = \frac{t}{4\log 3} \quad => \quad x = \frac{\sqrt{t}}{2\sqrt{log\ 3}}$$

diff. w.r.t x,

$$dx = \frac{t^{-1/2}}{4\sqrt{log\ 3}} dt \qquad \text{lim}\longrightarrow[\ 0, \infty\ ]$$

$$\therefore \text{I} = \int_0^\infty \frac{e^{-t}}{4\sqrt{\log 3}}\ t^{-1/2}$$

$$\therefore \text{I} = \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t}.t^{-1/2}\ \text{dt}$$

$$\boxed{\therefore \text{I} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}} \qquad \text{..........}\{\ \int_0^\infty e^{-t}.t^{-1/2}\ dt = \sqrt{\pi}\ \}$$

(b) Solve $(2y^2 - 4x + 5)dx = (y - 2y^2 - 4xy)dy$ **[3]**

Ans : $(2y^2 - 4x + 5)dx = (y - 2y^2 - 4xy)dy$

Compare with Mdx + Ndy = 0

$\therefore$ M = $(2y^2 - 4x + 5)$ $\qquad$ $\therefore$ N = - $(y - 2y^2 - 4xy)$

$\frac{\partial M}{\partial y} = 4y$ $\qquad$ $\frac{\partial N}{\partial x} = 4y$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given diff. eqn is exact .

The solution of exact diff. eqn is given by ,

$$\int M dx + \int [N - \frac{\partial}{\partial y} M dx] \, dy = c$$

$$\int M dx = \int((2y^2 - 4x + 5))dx = 2xy^2 - 2x^2 + 5x$$

$$\frac{\partial}{\partial y} \int M dx = 4xy$$

$$\int [N - \frac{\partial}{\partial y} M dx] \, dy = \int [4xy - y + 2y^2 - 4xy]dy = \frac{2}{3}y^3 - \frac{y^2}{2}$$

$$\boxed{\therefore\ 2xy^2 - 2x^2 + 5x + \frac{2}{3}y^3 - \frac{y^2}{2} = c}$$

(c) Solve the ODE $(D-1)^2(D^2+1)^2 y = 0$ [3]

Ans : $(D-1)^2(D^2+1)^2 y = 0$

For complementary solution ,

$$f(D) = 0$$

$$(D-1)^2(D^2+1)^2 = 0$$

$\therefore (D-1)^2 = 0$ $\therefore (D^2+1)^2 = 0$

$D - 1 = 0$ *for two times* $(D^2+1) = 0$ *for two times*

$\therefore$ D -1=0 $\therefore D^2 = -1$

Roots are : D = 1,1,+i,+i,-i,-i

$$\boxed{\therefore\ y_c = (c_1 + xc_2)e^x + [(c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x]}$$

(d) Evaluate $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$ [3]

Ans : let $I = \int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$

$$= \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right]_0^{x^2} dx$$

$$= \int_0^1 \frac{(e^x - 1)}{\frac{1}{x}}\, dx$$

$$= \int_0^1 x.e^x\, dx - \int_0^1 x.\, dx$$

$$= [\, x.e^x - e^x\,]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= e - e + 1 - \frac{1}{2}$$

$$\boxed{\therefore I = \frac{1}{2}}$$

(e) Evaluate $\int_0^1 \frac{x^a - 1}{log\, x}\, dx$ [4]

Ans : let $I = \int_0^1 \frac{x^a - 1}{\log x}\, dx$

Taking 'a' as parameter ,

$$I(a) = \int_0^1 \frac{x^a - 1}{\log x}\, dx \qquad \text{-------- (1)}$$

differentiate w.r.t a ,

$$\frac{dI(a)}{da} = \frac{d}{da} \int_0^1 \frac{x^a - 1}{\log x}\, dx$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - 1}{\log x}\, dx \qquad \text{.........\{ D.U.I.S f(x)\}}$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{x^a .\log x}{\log x}\, dx \qquad \text{.........}\left\{ \frac{dx^a}{da} = x^a . \log a \right\}$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 x^a\, dx$$

$\therefore \frac{dI(a)}{da} = \left[\frac{x^{a+1}}{a+1} \right]_0^1$

$\therefore \frac{dI(a)}{da} = \frac{1}{a+1} - 0$

$\therefore \frac{dI(a)}{da} = \frac{1}{a+1}$

now , integrate w.r.t a,

$I(a) = \int \frac{1}{a+1}\, da$

$I(a) = \log(a+1) + c$ -------- (2)

where c is constant of integration

put a=0 in eqn (1),

$I(0) = \int_0^1 0\, dx = 0$

And

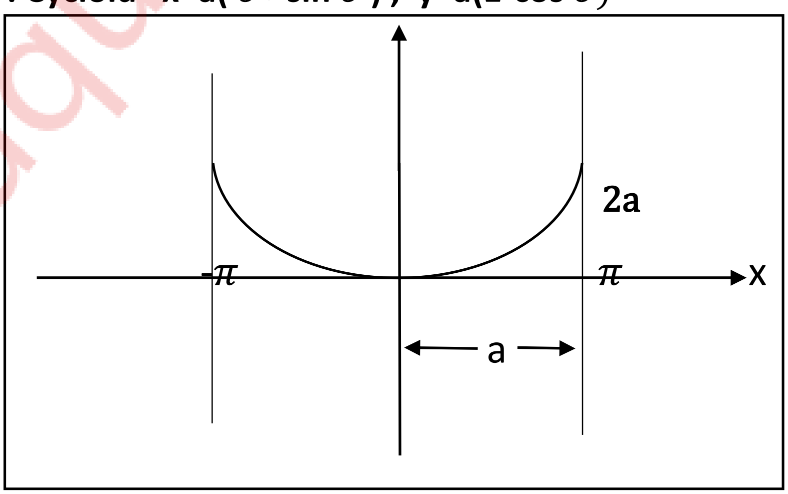
From eqn (2), $I(0)= c$

$\therefore c = 0$

$\boxed{\therefore I = \log(a+1)}$

(f) Find the length of cycloid from one cusp to the next , where $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$. **[4]**

Ans : Given curve : Cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$



The length of given curve is :

$$S = \int_{\theta_1}^{\theta_2} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta$$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore\ (\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = a^2[\,1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\,]$$

$$= 2a^2[\,1 + \cos\theta]$$

$$= 4a^2[\,\cos^2 \theta/2]$$

$$\therefore \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} = 2a\cos\theta/2$$

$$\therefore S = \int_{-\pi}^{\pi} 2a\cos\theta/2\, d\theta$$

$$= 2\times\int_0^{\pi} 2a\cos\theta/2\, d\theta$$

$$= 4a\ [\,2\sin\theta/2\,]_0^{\pi}$$

$$\boxed{\therefore S = 8a}$$

Q.2.(a) Solve $(D^2 - 3D + 2)y = 2e^x\sin(\frac{x}{2})$ **[6]**

Ans : $(D^2 - 3D + 2)y = 2e^x\sin(\frac{x}{2})$

For complementary function ,

$$f(D) = 0$$

$$\therefore (D^2 - 3D + 2) = 0$$

Roots are : D = 2,1 Real roots .

$$\boxed{y_c = c_1e^x + c_2e^{2x}}$$

For particular integral ,

$$y_p = \frac{1}{f(D)}X$$

$$= \frac{1}{(D^2-3D+2)}\ 2e^x sin(\frac{x}{2})$$

$$= \ 2e^x \frac{1}{(D+1)^2-3(D+1)+2}\ sin(\frac{x}{2})$$

$$= \ 2e^x \frac{1}{(D^2-D)} sin(\frac{x}{2})$$

$$= \ 2e^x \frac{1}{-\left(\frac{1}{4}\right)-D} sin(\frac{x}{2})$$

$$= -8e^x \ \frac{1}{4D+1} sin(\frac{x}{2})$$

$$= \ -8e^x \ \frac{4D-1}{16D^2-1} sin(\frac{x}{2})$$

$$\boxed{y_p = \ \frac{8}{5}e^x(-sin\left(\frac{x}{2}\right) - 2cos(\frac{x}{2}))}$$

The general solution of given diff. eqn is given by,

$$\boxed{y_c = y_c + y_p = c_1e^x + c_2e^{2x} + \frac{8}{5}e^x(-sin\left(\frac{x}{2}\right) - 2cos(\frac{x}{2}))}$$

(b) Using D.U.I.S prove that $\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})}dx = \frac{\sqrt{\pi}}{2}e^{-2a}, a > 0$ **[6]**

Ans : Let I(a) = $\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})}dx$ ....(1)

Taking 'a' as parameter diff. w.r.t. a,

$$\frac{dI(a)}{da} = \frac{d}{da}\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})}dx$$

Apply D.U.I.S rule ,

$$\frac{dI(a)}{da} = \int_0^\infty \frac{\partial}{\partial a} e^{-(x^2+\frac{a^2}{x^2})}dx$$

$$= \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} . \frac{-2a}{x^2}\ dx$$

Put $\frac{a}{x} = t$, $\frac{-a}{x^2} dx = dt$

Limits [$\infty, 0$**]**

$$\frac{dI(a)}{da} = \int_\infty^0 e^{-\left(t^2+\frac{a^2}{t^2}\right)} . 2dt = -2\int_0^\infty e^{-\left(t^2+\frac{a^2}{t^2}\right)} dt = -2I(a)$$

$$\frac{dI(a)}{da} = -2I(a)$$

$$\therefore \frac{dI(a)}{I(a)} = -2da$$

Integrating both sides ,

log [I(a)]=-2a +log c

I(a) = c.e^{-2a}

put a=0 in above eqn and eqn (1)

$\therefore$ **I(a) = c =** $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ **..........{ Using gamma function }**

$$\boxed{\therefore I(a) = \frac{\sqrt{\pi}}{2} e^{-2a}}$$

(c) Change the order of integration and evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ **[8]**

Ans : Let I = $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$

Region of integration is : $x \le y \le \sqrt{2-x^2}$

$0 \le x \le 1$

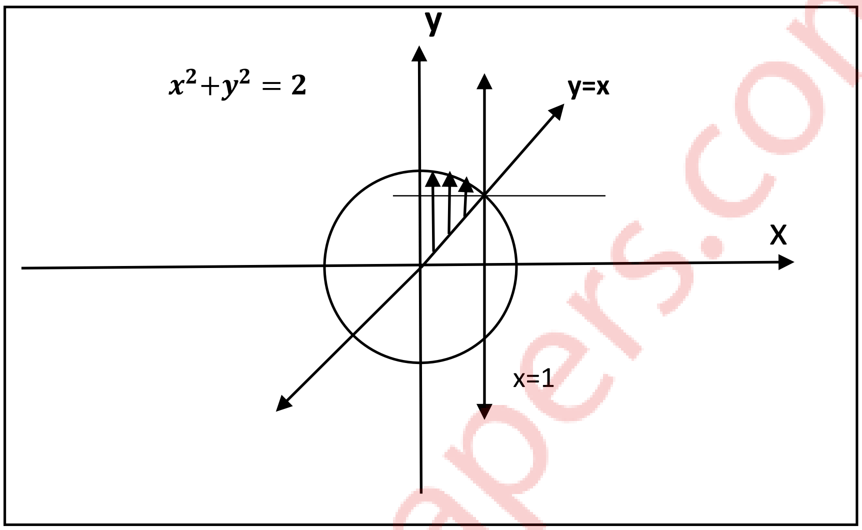
Curves : (i) y = x line

(ii) x=0 , x=1 lines parallel to the y axis .

(iii) $y = \sqrt{2 - x^2}$ => $x^2 + y^2 = 2$

Circle with centre (0,0) and radius $\sqrt{2}$.

Intersection of circle and y = x line is (1,1) in 1st quadrant.



Divide the region into two parts as shown in fig.

After changing the order of integration :

For one region : $0 \le x \le y$

$0 \le y \le 1$

For another region : $0 \le x \le \sqrt{2 - y^2}$

$1 \le y \le \sqrt{2}$

$$\therefore I = \int_0^1 \int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_0^1 [\sqrt{x^2+y^2}]_0^y \, dy + \int_1^{\sqrt{2}} [\sqrt{x^2+y^2}]_0^{\sqrt{2-y^2}}$$

$$= \int_0^1 (\sqrt{2}.y - y)\,dy + \int_1^{\sqrt{2}} (\sqrt{2} - 1)\,dy$$

$$= (\sqrt{2} - 1)[\tfrac{y^2}{2}]_0^1 + [\sqrt{2}y - \tfrac{y^2}{2}]_1^{\sqrt{2}}$$

$= \quad 1 - \frac{1}{\sqrt{2}}$

$$\boxed{\therefore I = \frac{\sqrt{2}-1}{\sqrt{2}}}$$

Q.3(a) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dxdydz$ **[6]**

Ans : Let I $= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dxdydz$

$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dzdydx$

$= \int_0^1 \int_0^{1-x} \left[\frac{1}{-2(x+y+z+1)^{-2}} \right]_0^{1-x-y}$ dy dx

$= -\int_0^1 \int_0^{1-x} \frac{1}{2} \left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2} \right] dy\, dx$

$= -\int_0^1 \frac{1}{2} \left[\frac{1}{4} y + \frac{1}{(x+y+1)^1} \right]_0^{1-x}$ dx

$= \int_0^1 \frac{1}{2} \left\{ \left[\frac{1}{4} (1-x) - \frac{1}{2} \right] + \left[\frac{1}{x+1} \right] \right\} dx$

$= \frac{1}{2} \left[\frac{1}{4} \left(\frac{(1-x)^2}{8} \right) - \frac{x}{2} + \log (x+1) \right]_0^1$

$$\boxed{\therefore I = \frac{1}{2} \left[\log 2 - \frac{5}{8} \right]}$$

(b) Find the mass of the lemniscate $r^2 = a^2 \cos 2\theta$ if the density at any point is Proportional to the square of the distance from the pole . [6]

Ans : Given curve : $r^2 = a^2 \cos 2\theta$ is lemniscate.

The density at any point is proportional to the square of dist. From the pole.

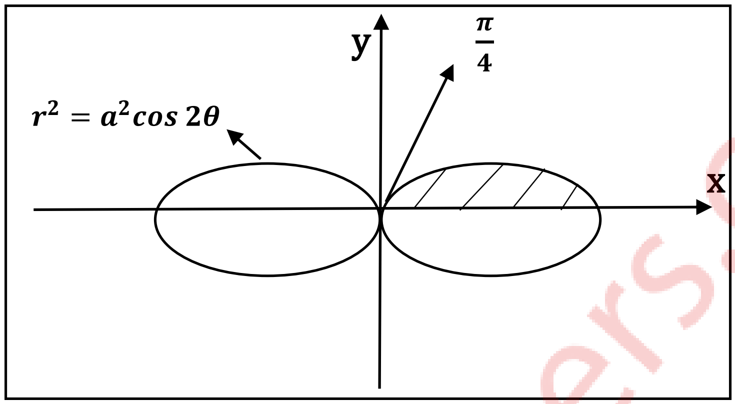
Distance from the pole = r

$\therefore Density \propto r^2$

$\therefore Density = k.r^2$

The mass of the lemniscate is given by ,

$$M = \int_{\theta_1}^{\theta_{21}} \int_{r_1}^{r_2} density\; r\; dr\; d\theta$$



$$\therefore \quad M = 4 \times \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} k.r^2.r dr d\theta$$

$$= 4k \times \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= k \times \int_0^{\frac{\pi}{4}} a^4.\cos^2 2\theta.d\theta$$

We can solve this definite integral by beta function.

Put $2\theta = t$ => $2\,d\theta = dt$

Limits $[\,0, \frac{\pi}{2}]$

$$\therefore \; M = ka^4 \int_0^{\frac{\pi}{2}} \cos^2 t.\frac{dt}{2}$$

$$= \frac{ka^4}{2} \times \frac{1}{2}\, \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$\boxed{\therefore \; M = \frac{ka^4\pi}{8}}$$

(c) Solve $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4\log x$ **[8]**

Ans : $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4\log x$

The given diff. eqn is Cauchy's homogeneous eqn .

Multiply the given eqn by x,

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4x\log x$$

Put x = e^z log x = z

Diff. w.r.t x,

$\frac{1}{x} = \frac{dz}{dx}$ but $\frac{dy}{dx} = \frac{dy}{dz} . \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$

$\therefore$ x $\frac{dy}{dx} = D$y

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$ where D $=\frac{d}{dz}$

$\therefore$ [D(D-1)(D-2)+3D(D-1)+D+1]y=4z. e^z

$\therefore$ [$D^3 + 1$]y=4z. e^z

For complementary solution ,

$$f(D) = 0$$

$\therefore$ [$D^3 + 1$] = 0

Roots are: D = -1 , $\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Roots of the eqn are real and complex.

$$\boxed{\therefore\ y_c = c_1e^{-z} + e^{z/2}(c_2 \cos\frac{\sqrt{3}z}{2} + c_3 \sin\frac{\sqrt{3}z}{2})}$$

For particular integral ,

$$y_p = \frac{1}{f(D)}\ X = \frac{1}{(D^3+1)}\ 4\ z.e^z$$

$$= 4e^z \frac{1}{(D+1)^3+1}\ z$$

$$= 4e^z \frac{1}{D^3+3D^2+3D+2}\ z$$

$$\therefore\ y_p = e^z(2z-3)$$

The general solution of given diff. eqn is ,

$$y_g = y_c + y_p = c_1e^{-z} + e^{z/2}\left(c_2\cos\frac{\sqrt{3}z}{2} + c_3\sin\frac{\sqrt{3}z}{2}\right) + e^z(2z-3)$$

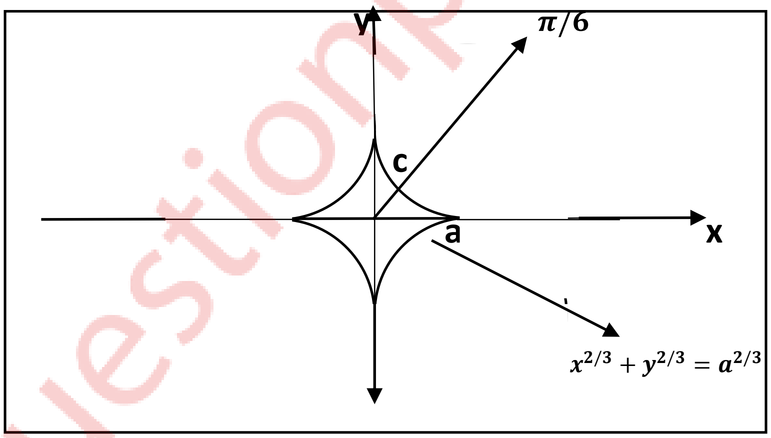
Resubstitute z,

$$\therefore y_g = \frac{c_1}{x} + \sqrt{x}\left(c_2\cos\frac{\sqrt{3}\log x}{2} + c_3\sin\frac{\sqrt{3}\log x}{2}\right) + x(2\log x - 3)$$

Q.4(a) Prove that for an astroid $x^{2/3} + y^{2/3} = a^{2/3}$, the line $\theta = \pi/6$ Divide the arc in the first quadrant in a ratio 1:3. **[6]**

Ans : Given curve : astroid $x^{2/3} + y^{2/3} = a^{2/3}$

The line $\theta = \pi/6$ cuts the asroid in 1 st quadrant.



C is the point on the curve which cuts the arc.

Length of astroid in first quadrant:

Put x= $a\cos^3 t$ and y=$a\sin^3 t$

dx=-3asin t.$\cos^2 t$dt dy=3acos t.$\sin^2 t$dt

$$S = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^{\pi/2} \sqrt{(-3a\sin t.\cos^2 t)^2 + (3a\cos t.\sin^2 t)^2}\ dt$$

$= \int_0^{\pi/2} 3a.\sin t.\cos t \, dt$

$= \frac{3}{2} a \int_0^{\pi/2} \sin 2t \, dt$

$= \frac{3}{4} a \left[- \cos 2t \right]_0^{\pi/2}$

$\boxed{\therefore S = \frac{3}{2} a}$ ....(1)

Now the length of the curve ac : Just put $\frac{\pi}{6}$ *insted of* $\frac{\pi}{2}$ because the curve is Only upto given line.

$\therefore$ S(ac) = $\int_0^{\pi/6} 3a \sin t \, . \cos t \, dt = \frac{3}{4} a \left[- \cos 2t \right]_0^{\pi/6}$

$= \frac{3}{4} a \left[- \frac{1}{2} + 1 \right]$

$\boxed{S(ac) = \frac{3}{8} a}$ ....(2)

Legnth of remaining part $= \frac{3}{2} a - \frac{3}{8} a = \frac{9}{8} a$ ....(3)

Divide eqn (3) and (2).

The line $\frac{\pi}{6}$ cuts the given astroid in the ratio of 1:3

Hence proved.

(b) Solve $(D^2 - 7D - 6)y = (1 + x^2)e^{2x}$ **[6]**

Ans : $(D^2 - 7D - 6)y = (1 + x^2)e^{2x}$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 - 7D - 6) = 0$$

Roots are : $D = \frac{7}{2} + \frac{\sqrt{73}}{2}$, $\frac{7}{2} - \frac{\sqrt{73}}{2}$

Roots of the given diff. eqn are irrational roots .

$$\boxed{y_c = e^{\frac{7x}{2}}(c_1 cosh\frac{\sqrt{73}}{2} + c_2 sinh\frac{\sqrt{73}}{2})}$$

For particular integral,

$$y_p = \frac{1}{f(D)} X$$

$$= \frac{1}{(D^2-7D-6)}[e^{2x} + e^{2x}x^2]$$

$$= \frac{1}{(D^2-7D-6)} e^{2x} + \frac{1}{(D^2-7D-6)} e^{2x}x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}\frac{1}{(D+2)^2-7(D+2)-6} x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}\frac{1}{D^2-3D-16} x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}[\frac{1}{-16}(\frac{1}{1+\frac{3D-D^2}{16}})\ ]x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}[\frac{1}{-16}(\frac{1}{1+\frac{3D-D^2}{16}})\ ]x^2$$

$$= -\frac{e^{2x}}{16}[\,1 + (1 + \frac{3D-D^2}{16})^{-1}x^2\,]$$

$$= -\frac{e^{2x}}{16}\{\ 1+ [1-\frac{3D-D^2}{16} + (\frac{3D-D^2}{16})^2\ ]\,x^2\ \}$$

$$= -\frac{e^{2x}}{16}\{\,1 + [x^2 - \frac{3}{8}x + \frac{2}{16} + \frac{9}{16\times 8}\ ]\,\}$$

$$= -\frac{e^{2x}}{16}\{\,1 + [x^2 - \frac{3}{8}x + \frac{25}{128}\ ]\,\}$$

$$\boxed{y_p = -\frac{e^{2x}}{16} - \frac{e^{2x}}{16}[x^2 - \frac{3}{8}x + \frac{25}{128}\ ]}$$

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c + y_p = e^{\frac{7x}{2}}(c_1 cosh\frac{\sqrt{73}}{2} + c_2 sinh\frac{\sqrt{73}}{2}) - \frac{e^{2x}}{16} - \frac{e^{2x}}{16}[x^2 - \frac{3}{8}x + \frac{25}{128}]}$$

(c) Apply Rungee Kutta method of fourth order to find an approximate Value of y when x=0.4 given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y = 1\ when\ x = 0$ Taking h=0.2. **[8]**

Ans : (I) $\frac{dy}{dx} = \frac{y-x}{y+x}$ $\quad x_0 = 0, y_0 = 1, h = 0.2$

$$f(x,y) = \frac{y-x}{y+x}$$

$$k_1 = h.f(x_0,y_0) = 0.2f(0,1) = 0.2$$

$$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2.f(0.1,1.1) = 0.1666$$

$$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2.f(0.1,1.0833) = 0.1661$$

$$k_4 = h.f(x_0 + h, y_0 + k_3) = 0.2f(0.2,1.1661) = 0.1414$$

$$k = \frac{k_1+2k_2+2k_3+k_4}{6} = \frac{0.2+2(0.1666)+2(0.1661)+0.1414}{6} = 0.1678$$

$$\boxed{\therefore\ y(0.2) = y_0 + k = 1 + 0.1678 = 1.1678}$$

(II) $x_1 = 0.2, y_2 = 1.1678, h = 0.2$

$$k_5 = h.f(x_1,y_1) = 0.2f(0.2,1.1678) = 0.1415$$

$$k_6 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_5}{2}\right) = 0.2.f(0.3,1.23855) = 0.1220$$

$$k_7 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_6}{2}\right) = 0.2.f(0.3,1.2285) = 0.1214$$

$$k_8 = h.f(x_1 + h, y_1 + k_7) = 0.2f(0.4,1.2892) = 0.1052$$

$$k* = \frac{k_5+2k_6+2k_7+k_8}{6} = \frac{0.1415+2(0.1220)+2(0.1215)+0.1052}{6} = 0.1222$$

$$\boxed{y(0.4) = y_1 + k* = 1.1678 + 0.1222 = 1.290}$$

Q.5(a) Use Taylor series method to find a solution of $\frac{dy}{dx} = xy + 1$, $y(0) = 0$

X=0.2 taking h=0.1 correct upto 4 decimal places. [6]

Ans : (I) $\frac{dy}{dx} = xy + 1$, $x_0 = 0, y_0 = 0,$ **h=0.1**

$$f(x,y) = 1 + xy$$

$y' = 1 + xy$ $\qquad y'_0 = 1$

$y'' = xy' + y$ $\qquad y''_0 = 0$

$y''' = xy'' + 2y'$ $\qquad y'''_0 = 2$

Taylor's series is given by ,

y(0.1) = $y_0 + h.y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots$

= 0 + 0.1(1) + 0 + $\frac{(0.1)^3}{6}(2)$

$\boxed{\text{y(0.1) = 0.1003}}$

(II) $x_1 = 0.1, y_1 = 0.1003$**,h=0.1**

$y' = 1 + xy$ $\qquad y'_0 = 1.01003$

$y'' = xy' + y$ $\qquad y''_0 = 0.201303$

$y''' = xy'' + 2y'$ $\qquad y'''_0 = 2.0401903$

$\therefore$ **y(0.2) = 0.1003+1.01003(0.1)+**$\frac{0.1^2}{2!}(0.201303) + \frac{0.1^3}{6}(2.0401903)$

$\boxed{\therefore \text{y(0.2) = 0.202708}}$

(b) Solve by variation of parameters $\left(\frac{d^2y}{dx^2} + 1\right)y = \frac{1}{1+\sin x}$ **[6]**

Ans : put $\frac{d}{dx} = D$

$$(D^2 + 1)y = \frac{1}{1+\sin x}$$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 + 1) = 0$$

Roots are : D $= i\,, -i$

Roots of given diff. eqn are complex.

The complementary solution of given diff. eqn is given by,

$$\boxed{\therefore\ y_c = c_1 \cos x + c_2 \sin x}$$

For particular solution ,

By method of variation of parameters,

$$y_p = y_1 p_1 + y_2 p_2$$

where $p_1 = \int \frac{-y_2 X}{w}\, dx$

$$p_2 = \int \frac{y_1 X}{w}\, dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$p_1 = \int \frac{-y_2 X}{w}\, dx = \int -\frac{\sin x}{1} \cdot \frac{1}{1+\sin x}\, dx = -\int \frac{\sin x}{1+\sin x} \frac{(1-\sin x)}{(1-\sin x)}\, dx$$

$$= -\int (\sec x . \tan x - \tan^2 x)\, dx$$

$$= -[\sec x - \tan x + x]$$

$$p_2 = \int \frac{y_1 . X}{w}\, dx = \int \frac{\cos x}{1} \frac{1}{1+\sin x}\, dx = \log(1 + \sin x)$$

$$y_p = -[\sec x - \tan x + x]\cos x + \log(1 + \sin x)\sin x$$

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c + y_p = c_1 cosx + c_2 sinx - [\text{secx} - \text{tanx} + \text{x}]\text{cosx} + log\,(1 + sinx)sinx}$$

(c) Compute the value of $\int_{0.2}^{1.4}(sin\ x - ln\ x + e^x)dx$ using (i) Trapezoidal Rule (ii) Simpson's (1/3)rd rule (iii) Simpson's (3/8)th rule by dividing Into six subintervals. **[8]**

Ans : let I = $\int_{0.2}^{1.4}(sin\ x - ln\ x + e^x)dx$

Dividing limits in six subintervals .

$\therefore$ n = 6 $\therefore$ h = $\frac{b-a}{n} = \frac{1.4-0.2}{6} = \frac{1}{5}$

$x_0 = 0.2$	$x_1 = 0.4$	$x_2 = 0.6$	$x_3 = 0.8$	$x_4 = 1.0$	$x_5 = 1.2$	$x_6 = 1.4$
$y_0 = 3.02$	$y_1 = 2.79$	$y_2 = 2.89$	$y_3 = 3.16$	$y_4 = 3.55$	$y_5 = 4.06$	$y_6 = 4.4$

(i) Trapezoidal rule : I = $\frac{h}{2}[X + 2R]$ -----------------(1)

$$X = sum\ of\ extreme\ ordinates = 7.42$$

$$R = sum\ of\ remaining\ ordinates = 16.45$$

I = $\frac{1}{5\times2}(7.42 + 2(16.45))$ ....(from 1)

$\boxed{\text{I} = 4.032}$

(ii) Simpson's $(1/3)^{rd}$ rule :

I = $\frac{h}{3}[X + 2E + 4O]$ ---------------(2)

$$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 4.4 + 3.02 = 7.42$$

$$E = sum\ of\ even\ base\ ordinates = y_2 + y_4 = 6.44$$

$$O = sum\ of\ odd\ base\ ordinates = y_1 + y_3 + y_5 = 10.01$$

I = $\frac{1}{3\times5}(7.42 + 2 \times 6.44 + 4 \times 10.01)$ ....(from 2)

$\boxed{I = 4.022}$

(iii) Simpson's $(3/8)^{th}$ rule :

$$I = \frac{3h}{8}[\,X + 2T + 3R\,] \qquad \text{-------------(3)}$$

$$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 4.4 + 3.02 = 7.42$$

$$T = sum\ of\ multiple\ of\ three\ base\ ordinates = y_3 = 3.16$$

$$R = sum\ of\ remaining\ ordinates = y_1 + y_2 + y_4 + y_5 = 13.49$$

$$\therefore I = \frac{3\times 1}{8\times 5}[\,7.42 + 2 \times 3.16 + 3 \times 13.49\,]$$

$$\boxed{\therefore I = 4.02075}$$

Q.6(a). Using beta functions evaluate $\int_0^{\pi/6} cos^6 3\theta.sin^2 6\theta d\theta$ [6]

Ans : let $I = \int_0^{\pi/6} cos^6 3\theta.sin^2 6\theta d\theta$

Put $3\theta = t$

Diff. w.r.t θ,

$d\theta = \frac{dt}{3}$ limits : $[0, \frac{\pi}{2}]$

$$\therefore I = \frac{1}{3}\int_0^{\pi/2} cos^6 t.sin^2 2t dt$$

$$= \frac{4}{3}\int_0^{\pi/2} cos^3 t\,(sin\,t.cos\,t)^2 dt$$

$$= \frac{4}{3}\int_0^{\pi/2} cos^5 t.sin^2 t.dt$$

$$= \frac{4}{3} \times \frac{1}{2} \times \beta(3, \frac{3}{2}) \qquad ..\{\int_0^{\pi/2} cos^m t.sin^n t.dt = \frac{1}{2} \times \beta(m+1, n+1)\}$$

$$\boxed{\therefore I = \frac{32}{315}}$$

(b) Evaluate $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} log(x^2+y^2)dxdy$ by changing to polar Coordinates. [6]

Ans : let I = $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} log(x^2+y^2)dxdy$

Region of integration : $y \le x \le \sqrt{a^2-y^2}$

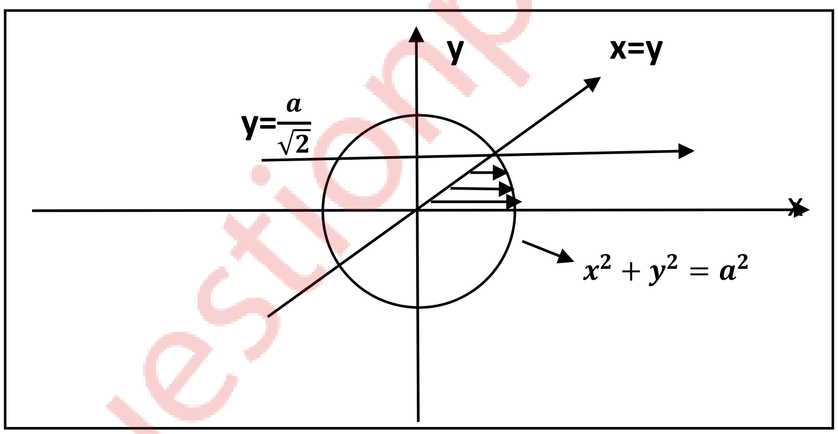
$0 \le y \le \frac{a}{\sqrt{2}}$

The line x=y is inclined at 45° to the +ve x-axis.

Curves : (i) x=y , y=0 , y=$\frac{a}{\sqrt{2}}$ lines

(ii) $x = \sqrt{a^2-y^2}$

$x^2+y^2 = a^2$ circle with centre (0,0) and radius a.



Cartesian coordinates ⟶ Polar coordinates

(x,y) ⟶ (r,θ)

Put x = r cos θ and y = r sin θ

$f(x,y) = log(x^2+y^2) = log\, r^2 = 2log\, r = f(r,\theta)$

Limits changes to : $0 \le r \le a$

$$0 \leq \theta \leq \frac{\pi}{4}$$

$\therefore$ I = $\int_0^{\frac{\pi}{4}} \int_0^{a} 2 log\, r . r\, dr\, d\theta$

= $2 \int_0^{\frac{\pi}{4}} [\, log\, r . \frac{r^2}{2} - \frac{r^2}{4}]_{0}^{a} d\theta$

= $2 \int_0^{\frac{\pi}{4}} \left[\, log\, a . \frac{a^2}{2} - \frac{a^2}{4}\right] d\theta$

$\boxed{\therefore I = \left[\, log\, a . \frac{a^2}{2} - \frac{a^2}{4}\right] \times \frac{\pi}{4}}$

(c) Evaluate $\int\int\int x^2 yzdxdydz$ over the volume bounded by planes x=0,y=0, z=0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ [8]

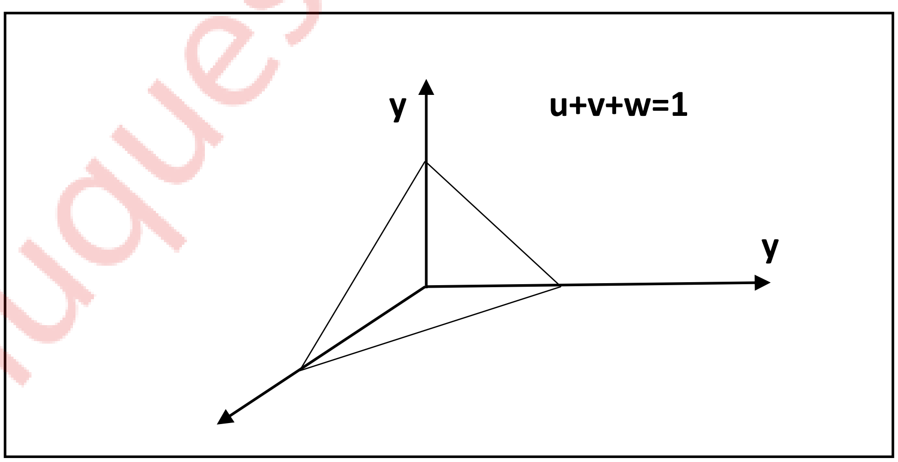
Ans : Let V = $\int\int\int x^2 dxdydz$

Region of integration is volume bounded by the planes x=0,y=0,z=0

And $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Put x = au , y = bv , z = cw

$\therefore$ dxdydz = abc du.dv



The intersection of tetrahedron with all axes is : (1,0,0),(0,1,0),(0,0,1).

$$0 \le w \le (1-u-v)$$

$$0 \le v \le (1-u)$$

$$0 \le u \le 1$$

The volume required is given by ,

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc\, a^2u^2bv.cw.dudvdw$$

$$= \frac{1}{2}a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2v\,(1-u-v)^2dvdu$$

$$= \frac{1}{2}a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2v[(1-u)^2 - 2(1-u)v + v^2]dudv$$

$$= \frac{1}{2}a^3b^2c^2 \int_0^1 u^2[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}]\begin{matrix} 1-u \\ 0 \end{matrix}\, du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \frac{u^2(1-u)^4du}{12}$$

$$= \frac{a^3b^2c^2}{24}\,\beta(3,5)$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{2!4!}{7!}\right)$$

$$\boxed{\therefore I = \frac{a^3b^2c^2}{2520}}$$

MUMBAI UNIVERSITY

SEMESTER 2

APPLIED MATHEMATICS SOLVED PAPER – DECEMBER 2017

N.B:- (1) Question no. 1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1.(a) Evaluate $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}}\, dx$ **[3]**

Ans : **Let** I = $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}}\, dx$

Put $x^2 = t$ => x = $\sqrt{t}$ => $\sqrt{x} = t^{1/4}$

Differentiate w.r.t x,

$\therefore$ dx = $\frac{1}{2\sqrt{t}}\, dt$ lim ⟶ [0,∞]

$\therefore \mathrm{I} = \int_0^{\infty} \frac{e^{-t}}{1} \frac{t^{-3/4}}{2}\, dt$

$\therefore \mathrm{I} = \frac{1}{2}\int_0^{\infty} e^{-t}.t^{\frac{1}{4}-1} dt$

But we know that ,

$\int_0^{\infty} e^{-t}.t^{n-1} dt = gamma(n)$

$\boxed{\therefore \mathrm{I} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)}$ ....{By the definition of gamma fn}

(b) Solve $(D^3 + 1)^2 y = 0$ **[3]**

Ans : $(D^3 + 1)^2 y = 0$

For complementary solution,

f(D) = 0

$\therefore (D^3 + 1)^2 = 0$

$$\therefore\ (D^3+1) = 0$$

Roots are : D = -1 , $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$ .. for two times

Roots of given diff. eqn are real and complex.

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c = (c_1+xc_2)e^{-x} + e^{\frac{x}{2}}[(c_3+xc_4)cosx + (c_5+xc_6)sinx]}$$

(c) Solve the ODE $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx + (x+xy^2)dy = 0$ [3]

Ans : Compare the given diff. eqn with Mdx + Ndy = 0

$\therefore$ M = $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)$ $\therefore$N = $(x+xy^2)$

$\frac{\partial M}{\partial y} = 1+y^2$ $\frac{\partial N}{\partial x} = 1+y^2$

$$\therefore\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential eqn is exact.

The solution of exact differential eqn is given by,

$$\int Mdx + \int[N - \frac{\partial}{\partial y}Mdx]\,dy = c$$

$$\int Mdx = \int\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx = xy+\frac{x}{3}y^3+\frac{x^3}{6}$$

$$\frac{\partial}{\partial y}\int Mdx = x+xy^2$$

$$\int[N - \frac{\partial}{\partial y}Mdx]\,dy = \int[\ x+xy^2-(x+xy^2)]dy = 0$$

$$\boxed{\therefore\quad xy+\frac{x}{3}y^3+\frac{x^3}{6} = c}$$

(d) Use Taylor's series method to find a solution of $\frac{dy}{dx}$ =1 + $y^2, y(0) = 0$

At x=0.1 taking h=0.1 correct upto 3 decimal places. [4]

Ans : $\frac{dy}{dx}$ =1 + y^2 $\quad x_0 = 0, y_0 = 0$, h=0.1

$$y' = 1 + y^2 \qquad y_0' = 1$$

$$y'' = 2yy' \qquad y_0'' = 0$$

$$y''' = 2yy'' + 2y'.y' \qquad y_0''' = 2$$

Taylor's series is given by :

y(0.1) = $y_0 + h.y_0' + \frac{h^2}{2!}y_0'' + \cdots$

= 0 + 0.1(1) + $\frac{0.1\times0.1}{2}(0)$ + $\frac{0.1\times0.1\times0.1}{6}(2)$

$\boxed{\text{y(0.1)} = 0.10033}$

(e) Given $\int_0^x \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})$ using DUIS find the value of

$\int_0^x \frac{dx}{(x^2+a^2)^2}$ [4]

Ans : $\int_0^x \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

Differentiate w.r.t a , taking 'a' as parameter

$\frac{d}{da}\int_0^x \frac{1}{x^2+a^2}dx$ = $\frac{d}{da}[\frac{1}{a}\tan^{-1}(\frac{x}{a})]$

Applying D.U.I.S rule,

D.U.I.S rule says that if function and its partial derivative is continuous then we can apply differential operator in the integral operator by converting it into partial derivative taking one parameter fro function.

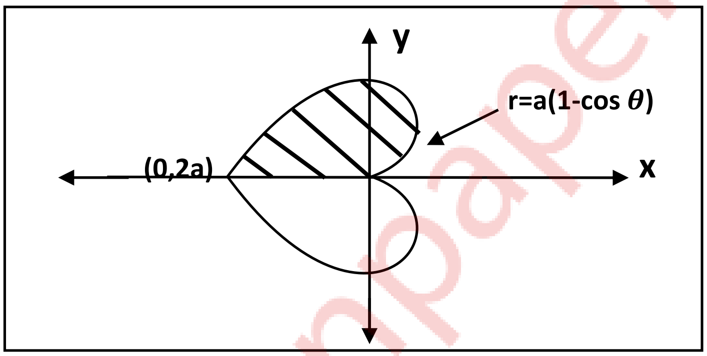
$$\int_0^x \frac{\partial}{\partial a}\frac{1}{x^2+a^2}\,dx = -\frac{1}{a}\tan^{-1}\frac{x}{a}\times\frac{1}{a} + \frac{-x}{a(x^2+a^2)}$$

$$\int_0^x \frac{2a^2}{x^2+a^2}\,dx = -\frac{1}{a}\tan^{-1}\frac{x}{a}\times\frac{1}{a} + \frac{-x}{a(x^2+a^2)}$$

$$\boxed{\int_0^x \frac{dx}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}}$$

(f) Find the perimeter of the curve r=a(1-cos θ) **[4]**

Ans : **Curve : r=a(1-cos θ)**



Perimeter of given curve is ,

$$S = 2\times\int_0^\pi \sqrt{r^2 + (\frac{dr}{d\theta})^2)}\,d\theta$$

$$\frac{dr}{d\theta} = a(\sin\theta) \quad => \quad (\frac{dr}{d\theta})^2 = a^2\sin^2\theta$$

$$r^2 + (\frac{dr}{d\theta})^2 = a^2[1 - 2\cos\theta + \cos^2\theta] + a^2\sin^2\theta$$

$$\sqrt{r^2 + (\frac{dr}{d\theta})^2} = \sqrt{2}a\,(1-\cos\theta)^{1/2}$$

$$= \sqrt{2}a\sqrt{2}\,\sin(\frac{\theta}{2})$$

$$\therefore \quad S = 2\int_0^\pi \sqrt{2}a\sqrt{2}\,\sin(\frac{\theta}{2})\,d\theta$$

$$= 4a\int_0^\pi \sin\left(\frac{\theta}{2}\right)d\theta$$

$= \; 4a\,[-2\cos(\frac{\theta}{2})\,]_{0}^{\pi}$

$\boxed{\therefore \quad S = 8a}$

Q.2.(a) Solve $(D^3+D^2+D+1)y = sin^2x$ **[6]**

Ans : $(D^3+D^2+D+1)y = \sin^2 x$

For complementary solution ,

$$f(D) = 0$$

$$\therefore\ (D^3+D^2+D+1) = 0$$

Roots are : D = -1 , +i , -i

The complementary solution of given diff eqn is ,

$$\boxed{y_c = c_1 cos\, x + c_2 sin\, x + c_3 e^{-x}}$$

For complementary solution ,

$$y_p = \frac{1}{f(D)}X = \frac{1}{(D^3+D^2+D+1)}\, sin^2x = \frac{1}{2(D^3+D^2+D+1)}(1 - cos2x)$$

$$= \frac{1}{2(D^3+D^2+D+1)}\, e^{0x} - \frac{1}{2(D^3+D^2+D+1)} cos\, 2x$$

$$= \frac{1}{2} - \frac{1}{2} \times \frac{1}{-D-4+D+1} cos\, 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{1}{D+1}\ cos\, 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{1}{D+1}\frac{D-1}{D-1}\ cos\, 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{D-1}{D^2-1}\ cos\, 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{-2sin\, 2x - cos\, 2x}{-4-1}\ cos\, 2x$$

$$\boxed{y_p = \frac{1}{2} + \frac{1}{30}.(2sin\, 2x + cos\, 2x)}$$

The general solution of given diff. eqn is given by,

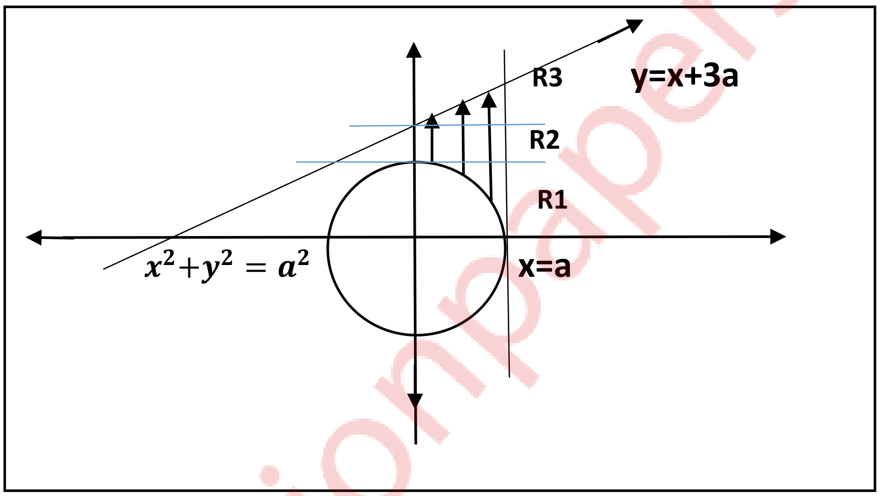
$$y_g = y_c + y_p = c_1 \cos x + c_2 \sin x + c_3 e^{-x} + \frac{1}{2} + \frac{1}{30}.(2\sin 2x + \cos 2x)$$

(b) Change the order of integration $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dxdy$ **[6]**

Ans : let I = $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dxdy$

Region of integration is : $\sqrt{a^2 - x^2} \le y \le x + 3a$

$$0 \le x \le a$$



Intersection of x=a and y=x+3a is (a,4a).

Intersection of x=0 and y=x+3a is (0,3a).

Divide the region into three parts R1,R2 and R3

$\therefore$ **R = R1** $\cup R2 \cup R3$

For region R1 : $\sqrt{a^2 - y^2} \le x \le$ a

$$0 \le y \le a$$

For region R2 : $0 \le x \le a$

$$a \le y \le 3a$$

For region R3 : $(y - 3a) \le x \le a$

$$3a \le y \le 4a$$

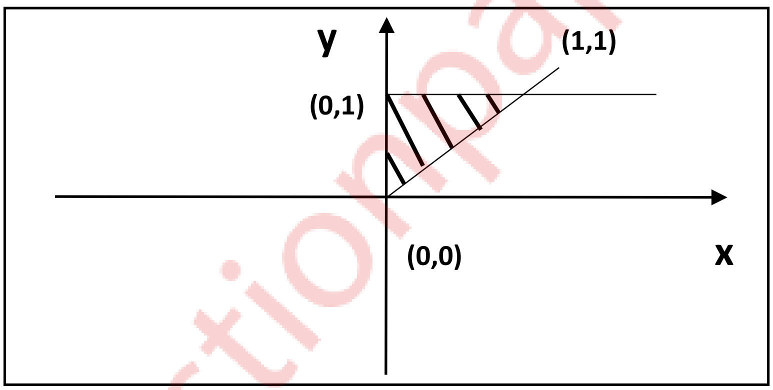
After changing the order of integration fro dydx to dxdy

$$\therefore I = \int_0^a \int_{\sqrt{a^2-y^2}}^{a} f(x,y)\,dxdy + \int_a^{3a}\int_0^a f(x,y)\,dxdy + \int_{3a}^{4a}\int_{(y-3a)}^{4a} f(x,y)\,dxdy$$

(c) Evaluate $\int\int \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}}\,dxdy$, where R is triangle whose vertices are (0,0),(1,1),(0,1). **[8]**

Ans : let $I = \int\int \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}}\,dxdy$

Region of integration : Triangle whose vertices are (0,0),(1,1),(0,1)



The equation of lines from diagram are : y=1,x=y

$$0 \le x \le y$$

$$0 \le y \le 1$$

$$\therefore I = \int_0^1 \int_0^y \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}}\,dxdy$$

$$= \int_0^1 \int_0^y \frac{2y^5.x}{\sqrt{(1-y^4)+x^2y^2}}\,dx\,dy$$

$$= \int_0^1 \int_0^y \frac{2y^5.x}{\sqrt{\frac{1-y^4}{y^2}+x^2}} . \frac{1}{y}\, dx\, dy$$

$$= \int_0^1 2y^4 [\sqrt{\frac{1-y^4}{y^2} + x^2}\ ]_0^y\, dy$$

$$= \int_0^1 2y^4 \left[\frac{1}{y} - \frac{\sqrt{1-y^4}}{y}\right] dy$$

$$= 2\int_0^1 [\, y^3 - \sqrt{1-y^4}\, .\, y^3]\, dy$$

$$= 2[\frac{y^4}{4} + \frac{1}{4} . \frac{(1-y^4)^{3/2}}{3/2}\,]_0^1$$

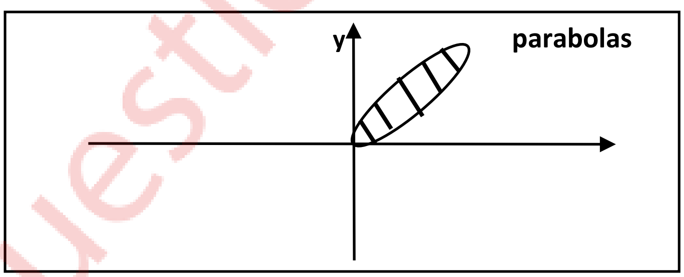
$$= 2[\ \frac{1}{4} - \frac{1}{4} . \frac{2}{3}]$$

$$\boxed{\therefore \mathrm{I} \ = \ \frac{1}{6}}$$

Q.3.(a) Find the volume enclosed by the cylinder $y^2 = x \; and \; y = x^2$

Cut off by the planes z=0,x+y+z=2. [6]

Ans : The solid is bounded by the parabolas $y^2 = x, x^2 = y$ in the x y plane .



In x-y-z plane x+y+z =2 is top base.

The volume between this curves is given by ,

V = $\int\int z\, dx\, dy = \int\int (2 - x - y)\, dxdy$

From the diagram we can conclude that the intersection point of both

Parabolas are (0,0),(1,1).

$$\therefore V = \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - x - y)\,dxdy$$

$$= \int_0^1 \left[2y - xy - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(2\sqrt{x} - x\sqrt{x} - \frac{x}{2}\right) - \left(2x^2 - x^3 - \frac{x^4}{2}\right)\right] dx$$

$$= \left[\frac{4x^{3/2}}{3} - \frac{2x^{\frac{5}{2}}}{5} - \frac{x^2}{4} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1$$

$$\boxed{\therefore V = \frac{11}{30}}$$

(b) Using Modified Eulers method ,find an approximate value of y At x=0.2 in two step taking h=0.1 and using three iteration Given that $\frac{dy}{dx}$=x+3y , y=1 when x=0. **[6]**

Ans : **(I)** $\frac{dy}{dx}$**=x+3y** $x_0 = 0, y_0 = 1, h = 0.1$

$y_1^{(0)} = y_0 + h.f(x_0, y_0)$ **= 1+ 0.1(3)=1.3**

$$y_1^{n+1} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^n)]$$

Iteration	x_1	y_1^n	$x_1 y_1^n$	y_1^{n+1}
0	0.1	1.3	4	1.35
1	0.1	1.35	4.15	1.3575
2	0.1	1.3575	4.1725	1.3587

$\boxed{y(0.1)=1.3587}$

(II) $x_1 = 0.1, y_1 = 1.3587$

$y_2^0 = 1.77631$

$$y_2^{n+1} = y_1 + \frac{h}{2}[f(x_2, y_2) + f(x_2, y_2^n)]$$

Iteration	x_2	y_2^n	$x_2 y_2^n$	y_2^{n+1}
0	0.2	1.77631	5.52893	1.8439

1	0.2	1.8439	5.7317	1.8540
2	0.2	1.8540	5.762	1.8556

$$\boxed{y(0.2)=1.8556}$$

(c) Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4cos\ (log(1+x))$ **[8]**

Ans : $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4cos\ (log(1+x))$

Put x+1 = v => $\frac{dv}{dx}=1$

$$\frac{dy}{dx}=\frac{dy}{dv}$$

The given eqn changes to ,

$$v^2\frac{d^2y}{dv^2}+v\frac{dy}{dv}+y=4cos\ log\ v$$

Now put log v = z ∴v=e^z

$$[D(D-1)+D+1]y=4cos\ z$$

$$\therefore\ (D^2+1)y=4cos\ z$$

For complementary solution ,

$$f(D)=0$$

$$\therefore (D^2+1)=0$$

Roots are : i,-i

The complementary solution of given diff. eqn is ,

$$\boxed{\therefore\ y_c=c_1cos\ z+c_2sin\ z}$$

For particular integral ,

$$y_p=\frac{1}{f(D)}\ X=\frac{1}{(D^2+1)}\ 4\ cos\ z=4\frac{z}{2}\ sin\ z=2\text{ z sin z}$$

$$\boxed{\therefore\ y_p=2\ z\ sin\ z}$$

The general solution of given diff. eqn is given by,

$$y_g = y_c + y_p = c_1 cos\, z + c_2 sin\, z + \ 2\, z\, sin\, z$$

Resubstitute z and v,

$$\boxed{y_g = c_1 cos\,[log(x+1)] + c_2 sin\,[log(1+x)] + \ 2\, log(1+x) sin\,[log(1+x)]}$$

Q.4.(a) Show that $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\, dx = a \frac{\sqrt{\pi}\, \Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}$ **[6]**

Ans : Let I = $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\, dx$

Put $x^3 = a^3 t \quad => \quad x = a t^{\frac{1}{3}}$

Diff. w.r.t. x ,

$$dx = \frac{a}{3}\, t^{-2/3} \quad dt$$

Limits becomes $\rightarrow [0, 1]$

$$I = \int_0^1 (t)^{3/2} . (1-t)^{3/2} . t^{-2/3} \frac{a}{3}\, dt$$

$$= \frac{a}{3} \int_0^1 t^{5/6} (1-t)^{3/2} dt$$

$$= \frac{a}{3}\, \beta(\frac{5}{6}, \frac{3}{2})$$

$$\boxed{I = a \frac{\sqrt{\pi}\, \Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}}$$ **…………{ from the definition of beta function}**

(b) Solve $(D^2+2)y = e^x cos\, x + x^2 e^{3x}$ **[6]**

Ans : $(D^2+2)y = e^x cos\, x + x^2 e^{3x}$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 + 2) = 0$$

Roots are : D $= \sqrt{2}i , -\sqrt{2}i$

Roots of given diff. eqn are complex.

The complementary solution of given diff. eqn is given by,

$$\boxed{\therefore\ y_c = c_1 cos\,\sqrt{2}x + c_2 sin\,\sqrt{2}x}$$

For particular integral ,

$$y_p = \frac{1}{f(D)}X = \frac{1}{D^2+1}e^x cos\,x + \frac{1}{D^2+1}x^2 e^{3x}$$

$$= e^x\,\frac{1}{(D+1)^2+1}cos\,x + \frac{1}{D^2+1}x^2 e^{3x}$$

$$= e^x\frac{1}{D^2+2D+3}cos\,x + e^{3x}\frac{1}{(D+3)^2+2}x^2$$

$$= e^x\,\frac{1}{2}\,\frac{D-1}{D^2-1}.cosx + e^{3x}\frac{1}{D^2+6D+11}x^2$$

$$= e^x\frac{1}{4}(sin\,x + cos\,x) + \frac{e^{3x}}{11}[1 + \frac{6D+D^2}{11}]^{-1}x^2$$

$$= e^x\frac{1}{4}(sin\,x + cos\,x) + \frac{e^{3x}}{11}[\,1 - \frac{6D+D^2}{11} + \frac{36D^2}{121} + ..]x^2$$

$$\boxed{\therefore\ y_p = e^x\frac{1}{4}(sin\,x + cos\,x) + \frac{e^{3x}}{11}[\,x^2 - \frac{12x}{11} + \frac{50}{121}]}$$

The general solution of given diff. eqn is,

$$\boxed{y_g = y_c + y_p = c_1 cos\sqrt{2}x + c_2 sin\sqrt{2}x + e^x\frac{1}{4}(sin\,x + cosx) + \frac{e^{3x}}{11}[\,x^2 - \frac{12x}{11} + \frac{50}{121}]}$$

(c) Use polar co ordinates to evaluate $\int\int\frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$ over yhe area

Common to circle $x^2+y^2 = ax\ and\ x^2 + y^2 = by, a > b > 0.$ [8]

Ans : Let I = $\int\int \frac{(x^2+y^2)^2}{x^2y^2}\, dx\, dy$

Region of integration is : Area common to the circle

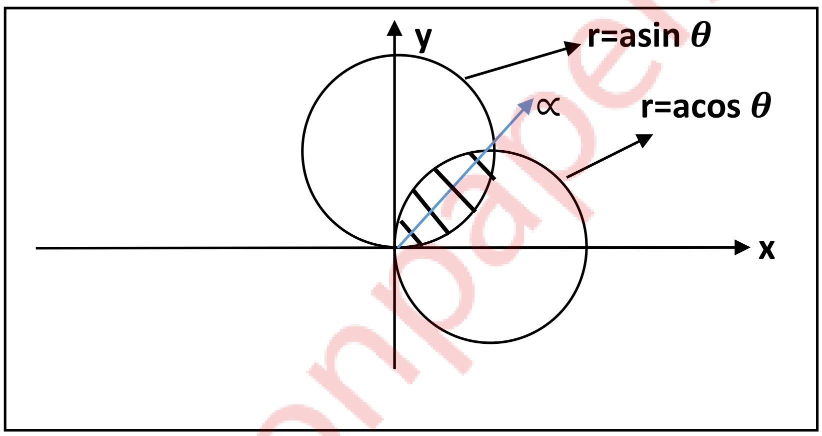
$$x^2+y^2 = ax \text{ and } x^2+y^2 = by$$

To change the Cartesian coordinates to polar coordinates

Put x= r cosθ and y = r sin θ

Circles : r=acos θ *and* r=asin θ

The function becomes : f(x,y)= $\frac{(x^2+y^2)^2}{x^2y^2} = \frac{r^4}{r^4 sin^2\theta\ cos^2\theta} = \frac{4}{sin^2 2\theta} = f(r,\theta)$



Intersection of both circles is at angle $=$ $\tan^{-1}\frac{a}{b}$.

Divide the region into two equal halves.

For one region , $0 \le r \le bsin\theta$

$0 \le \theta \le \alpha$

For another region , $0 \le r \le acos\,\theta$

$\alpha \le \theta \le \frac{\pi}{2}$

$$\therefore I = \int_0^{\alpha}\int_0^{bsin\theta} \frac{4rdrd\theta}{sin^2 2\theta} + \int_0^{acos\theta}\int_{\alpha}^{\frac{\pi}{2}} \frac{4rdrd\theta}{sin^2 2\theta}$$

$$\therefore I = \int_0^{\alpha} \frac{4}{sin^2 2\theta}\left[\frac{r^2}{2}\right]_0^{bsin\theta} d\theta + \int_0^{\frac{\pi}{2}} \frac{4}{sin^2 2\theta}\left[\frac{r^2}{2}\right]_0^{acos\theta} d\theta$$

$$= \frac{1}{2}b^2 \int_0^{\propto} sec^2\theta d\theta + \frac{a^2}{2}\int_{\alpha}^{\frac{\pi}{2}} cosec^2\theta d\theta$$

$$= \frac{1}{2} b^2 tan \propto + \frac{a^2}{2} cot \propto$$

$$= \frac{ab}{2} + \frac{ab}{2}$$

$$\boxed{\therefore I = ab}$$

Q.5.(a) Solve $ydx + x(1 - 3x^2y^2)dy = 0$ **[6]**

Ans : $ydx + x(1 - 3x^2y^2)dy = 0$ ....(1)

Compare the given eqn with Mdx + Ndy=0

$\therefore M = y$ $\quad\quad$ $\therefore N = x(1 - 3x^2y^2)$

$\frac{\partial M}{\partial y} = 1$ $\quad\quad$ $\frac{\partial N}{\partial x} = 1 - 9x^2y^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence the given diff. eqn is not exact.

But the given diff. eqn is in the form of

yf(xy)dx + xf(xy)dy = 0

Integrating factor = $\frac{1}{Mx-Ny} = \frac{1}{xy-xy+3x^3y^3} = \frac{1}{3x^3y^3}$

Multiply the I.F. to eqn (1),

$$\frac{1}{3x^3y^2} dx + \left[\frac{1}{3x^2y^3} - \frac{1}{y}\right] dy = 0$$

$\therefore M_1 = \frac{1}{3x^3y^2}$ $\quad\quad$ $N_1 = \left[\frac{1}{3x^2y^3} - \frac{1}{y}\right]$

Now this diff. eqn is exact.

The solution of given diff. eqn is given by,

$$\int Mdx + \int [N - \frac{\partial}{\partial y} Mdx] dy = c$$

$$\int M_1 dx = \int \frac{1}{3x^3y^2} dx = \frac{-1}{6y^2x^2}$$

$$\frac{\partial}{\partial y}\int M_1 dx = \frac{1}{3x^2y^3}$$

$$\int [\, N_1 - \frac{\partial}{\partial y}\int M_1 dx \,]\, dy = \int \left[\frac{1}{3x^2y^3} - \frac{1}{y} - \frac{1}{3x^2y^3}\right] dy$$

$$= \int \frac{-1}{y} dy == -\log y$$

$$\boxed{\therefore \ \frac{-1}{6y^2x^2} - \log y = c}$$

(b) Find the mass of a lamina in the form of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

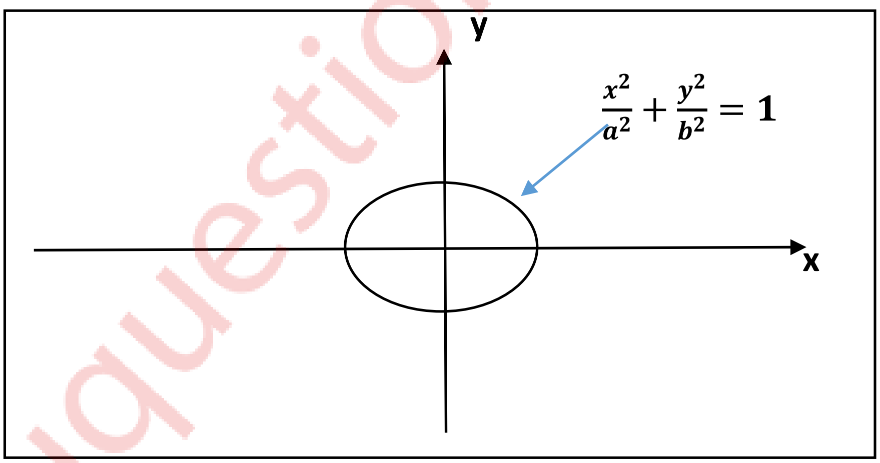
If the density at any point varies as the product of the distance from the

The axes of the ellipse. [6]

Ans : Mass of lamina is given by , M = $\int\int r\, dx\, dy$

r is the density function r =k xy

Ellipse eqn is : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$0 \le y \le b\sqrt{a^2 - x^2}/a$$

$$0 \le x \le a$$

$$\therefore M = 4\int_0^a \int_0^{b\sqrt{a^2-x^2}/a} kxy\, dy\, dx$$

$$= 4k\int_0^a x.\left[\frac{y^2}{2}\right]_0^{b\sqrt{a^2-x^2}/a} dx$$

$$= 2k\int_0^a x.\frac{b^2}{a^2}(a^2-x^2)dx$$

$$= \frac{2kb^2}{a^2}\int_0^a [a^2x - x^3]dx$$

$$= \frac{2kb^2}{a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$\boxed{\therefore\ M = \frac{ka^2b^2}{2}}$$

(c) Compute the value of $\int_0^{\frac{\pi}{2}} \sqrt{\sin x + \cos x}\,dx$ using (i)Trapezoidal rule (ii)Simpson's (1/3)rd rule (iii)Simpson's (3/8)th rule by dividing into six Subintervals. **[8]**

Ans : **Let I =** $\int_0^{\frac{\pi}{2}} \sqrt{\sin x + \cos x}\,dx$

Dividing limits into 6 subintervals . n=6

a=0 , b=$\frac{\pi}{2}$ $\therefore$**h=**$\frac{b-a}{n} = \frac{\pi}{12}$

$x_0 = 0$	$x_1 = \pi/12$	$x_2 = 2\pi/12$	$x_3 = 3\pi/12$	$x_4 = 4\pi/12$	$x_5 = 5\pi/12$	$x_6 = 6\pi/12$
$y_0 = 1$	$y_1 = 1.1067$	$y_2 = 1.1688$	$y_3 = 1.1892$	$y_4 = 1.1688$	$y_5 = 1.1067$	$y_6 = 1$

(i) Trapezoidal rule : **I =** $\frac{h}{2}[X + 2R]$ **-----------------(1)**

$$X = sum\ of\ extreme\ ordinates = 2$$

$$R = sum\ of\ remaining\ ordinates = 5.7402$$

I = $\frac{\pi}{12\times2}(2 + 2(5.7402))$ **……………….(from 1)**

$$\boxed{I = 1.7636}$$

(ii) Simpson's $(1/3)^{rd}$ **rule :**

I = $\frac{h}{3}[\,X + 2E + 4O\,]$ ---------------(2)

$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 1 + 1 = 2$

$E = sum\ of\ even\ base\ ordinates =\ y_2 + y_4 = 2.3376$

$O = sum\ of\ odd\ base\ ordinates =\ y_1 + y_3 + y_4 = 3.4026$

I = $\frac{\pi}{3\times 12}(2 + 2\times 2.3376 + 4\times 3.4026)$ ....(from 2)

$\boxed{\text{I = 1.7693}}$

(iii) Simpson's $(3/8)^{th}$ rule :

I = $\frac{3h}{8}[\,X + 2T + 3R\,]$ -------------(3)

$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 0 + 0.5 = 2$

$T = sum\ of\ multiple\ of\ three\ base\ ordinates =\ y_3 = 1.1892$

$R =\ sum\ of\ remaining\ ordinates\ =\ y_1 + y_2 + y_4 + y_5 = 4.551$

$\therefore$ I = $\frac{3\times\pi}{8\times 12}[\,2 + 2\times 1.1892 + 3\times 4.551\,]$

$\boxed{\therefore \text{I} = 1.7702}$

Q.6(a)Change the order of Integration and evaluate $\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4 - 4y^2}}\,dxdy$**[6]**

Ans : Let I = $\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4-4y^2}}\,dxdy$

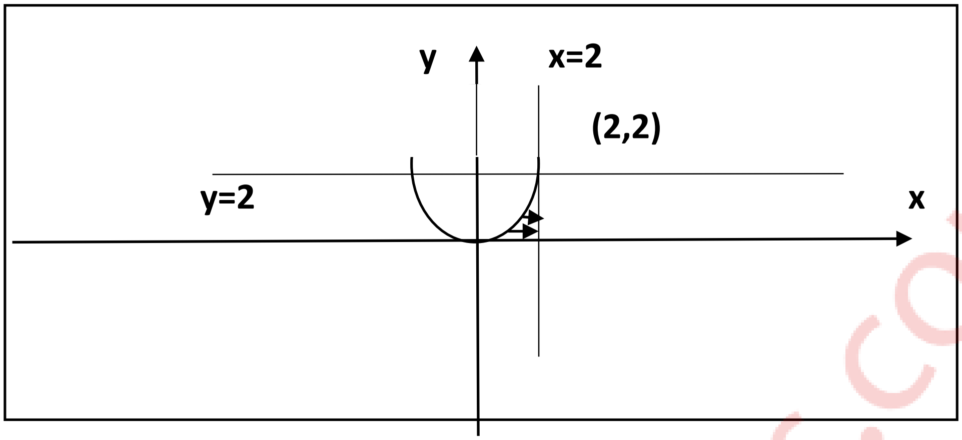
Region of integration : $\sqrt{2y} \le x \le 2$

$0 \le y \le 2$

Curves : (i) x = 2 , y = 2 , y = 0 are lines.

(ii) x = $\sqrt{2y}$ => $x^2 = 2y$

Parabola with vertex (0,0) opening in upward direction.



After changing the order of integration:

$$0 \le y \le \frac{x^2}{2}$$

$$0 \le x \le 2$$

$$\therefore \ \mathrm{I} = \int_0^2 \int_0^{\frac{x^2}{2}} \frac{x^2}{\sqrt{x^4 - 4y^2}} dydx$$

$$= \frac{1}{2} \int_0^2 \int_0^{\frac{x^2}{2}} \frac{x^2}{\sqrt{\frac{x^4}{4} - y^2}} \mathrm{dydx}$$

$$= \frac{1}{2} \int_0^2 x^2 [\sin^{-1}(\frac{y}{x^2/2})]_0^{\frac{x^2}{2}} \mathrm{dy}$$

$$\therefore \ \mathrm{I} = \frac{1}{2} \int_0^2 x^2 \frac{\pi}{2} \, dx$$

$$= \frac{\pi}{4} [\frac{x^3}{3}]_0^2$$

$$\boxed{\therefore \ \mathrm{I} = \frac{2\pi}{3}}$$

(b) Evaluate $\int \int \int x^2 dxdydz$ over the volume bounded by planes x=0,y=0 z=0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. [8]

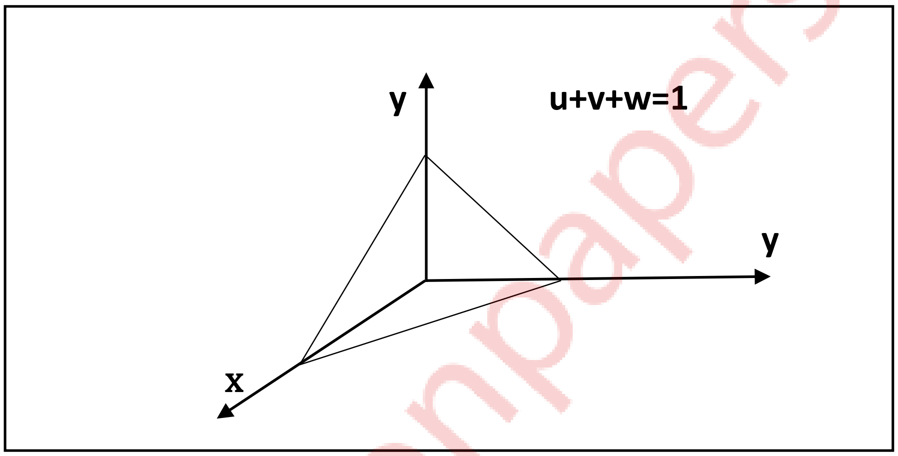
Ans : Let V = $\int \int \int x^2 dxdydz$

Region of integration is volume bounded by the planes x=0,y=0,z=0

And $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Put x = au , y = bv , z = cw

∴ dxdydz = abc du.dv.dw



The intersection of tetrahedron with all axes is : (1,0,0),(0,1,0),(0,0,1).

$$0 \le w \le (1-u-v)$$

$$0 \le v \le (1-u)$$

$$0 \le u \le 1$$

The volume required is given by ,

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc\, a^2u^2\, dudvdw$$

$$= a^3bc \int_0^1 \int_0^{1-u} (1-u-v)\, u^2 dvdu$$

$$= a^3bc \int_0^1 u^2 \left[v - uv - \frac{v^2}{2} \right]_0^{1-u} du$$

$$= a^3bc\int_0^1 u^2[\,1-u-u+u^2-\frac{u^2(1-u)^2)}{2}\,]\,du$$

$$= a^3bc\,[\,\frac{u^3}{3}-\frac{u^4}{2}+\frac{u^5}{5}-\frac{1}{2}(\frac{u^3}{3}-\frac{1}{2}u^4+\frac{u^5}{5})\,]_0^1$$

$$\boxed{\therefore V = \frac{1}{60}(a^3bc)}$$

(c) Solve by method of variation of parameters : $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$ [8]

Ans : $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$

For complementary solution ,

$$f(D)=0$$

$$\therefore\ (D^2-6D+9)=0$$

Roots are : D = 3 , 3 Real roots but repeatative.

The complementary solution of given diff. eqn is ,

$$\boxed{\therefore\ y_c = (c_1+xc_2)e^{3x}}$$

For particular solution ,

By method of variation of parameters,

$$y_p = y_1p_1+y_2p_2$$

where $p_1 = \int\frac{-y_2\,X}{w}\,dx$

$$p_2 = \int\frac{y_1\,X}{w}\,dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix} = e^{6x}$$

$$p_1 = \int\frac{-y_2\,X}{w}\,dx = \int\frac{xe^{3x}}{e^{6x}}\cdot\frac{e^{3x}}{x^2}\,\text{dx} = \int\frac{-1}{x}\,dx = -\log x$$

$$p_2 = \int \frac{y_1 X}{w}\, dx = \int \frac{e^{3x}}{e^{6x}} . \frac{e^{3x}}{x^2} dx = \int \frac{1}{x^2}\, dx = \frac{-1}{x}$$

The particular integral of given diff. eqn is given by,

$$\boxed{\therefore\ y_p = -e^{3x} \log x - e^{3x} = -e^{3x}(\log x + 1)}$$

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c + y_p = (c_1 + x c_2) e^{3x} - e^{3x}(\log x + 1)}$$

MUMBAI UNIVERSITY

SEMESTER – 2

APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1 a) Evaluate $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}dx$. **[3]**

ANS: $I = \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}dx.$

Put $x^3 = t$

$\therefore x = t^{\frac{1}{3}}$

$dx = \frac{1}{3}t^{\frac{-2}{3}}$

$\therefore I = \int_0^\infty e^{-t}.t^{-\frac{1}{6}}.\frac{1}{3}.t^{\frac{-2}{3}}\,dt$

$\therefore I = \int_0^\infty e^{-t}.t^{-\frac{5}{6}}\,dt$

$$\boxed{\therefore \boldsymbol{I} = \frac{1}{3}\left|\overline{\frac{1}{6}}\right..}$$

b) Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from y = 1 to y = 2. **[3]**

ANS: We have $x = \frac{y^3}{3} + \frac{1}{4y}$

Diff w.r.t. y, we get

$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(y^2-\frac{1}{4y^2}\right)^2 = y^4+\frac{1}{2}+\frac{1}{16y^4} = \left(y^2+\frac{1}{4y^2}\right)^2$$

We know that,

$$s = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$s = \int_1^2 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\, dy$$

$$s = \left[\frac{y^3}{3}-\frac{1}{4y}\right]_1^2$$

$$s = \frac{8}{3}-\frac{1}{8}-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$\boxed{s = \frac{59}{24}}$$

c) Solve $(D^2+D)y = e^{4x}$. **[3]**

ANS: For auxiliary equation,

$D^2+D = 0$

Solving we get,

D = -1, 0.

$\therefore$ C.F. = $C_1e^{-x}+C_2e^{0x}$

$\therefore C.F = C_1e^{-x}+C_2$

For P.I.,

$$y = \frac{e^{4x}}{D^2+D}$$

Now, put D = 4

$$\therefore y = \frac{e^{4x}}{4^2+4} = \frac{e^{4x}}{20}$$

$\therefore$ The complete solution is,

$$\boxed{y = C_1e^{-x}+C_2+\frac{e^{4x}}{20}.}$$

d) Evaluate $\int_0^1 \int_{x^2}^{x} xy(x+y)dydx$. **[3]**

ANS: We have,

$$I = \int_0^1 \int_{x^2}^{x} xy(x+y)dydx.$$

$$I = \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$I = \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$I = \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$I = \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$\boxed{\boldsymbol{I = \frac{3}{56}}}$$

e) Solve $(4x+3y-4)dx + (3x-7y-3)dy = 0$. **[4]**

ANS: Given, $(4x+3y-4)dx + (3x-7y-3)dy = 0$.

$\therefore$ M = $(4x+3y-4)$ and N = $(3x-7y-3)$

Differentiating M by y and N by x, we get,

$\frac{dM}{dy} = 3$ And $\frac{dN}{dx} = 3$

$\therefore \frac{dM}{dy} = \frac{dN}{dx}$

$\therefore$The given equations are exact.

For solution,

$\int M\ dx = \int (4x+3y-4)\ dx$

$\int M\ dx = 2x^2 + 3xy - 4x$

$\int (Term\ is\ N\ free\ from\ x) = \int -7y - 3\ dy$

$= \frac{-7y^2}{2} - 3y$

$\therefore$ The final solution is,

$$2x^2 + 3xy - 4x - \frac{7y^2}{2} - 3y = c$$

$$\boxed{4x^2 + 6xy - 8x - 7y^2 - 6y = c}$$

f) Solve $\frac{dy}{dx} = 1 + xy$ with initial condition $x_0 = 0,\ y_0 = 0.2$ By Taylors series method. Find the approximate value of y for x = 0.4(step size = 0.4).

ANS: The Taylor series is given by,

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' \ldots.. \qquad \ldots\ldots.. (1)$$

With $x_0 = 0, y_0 = 0.2, x = 0.4$

Now, y' = 1 + xy $\quad \therefore y_0' = 1$

y'' = y + xy' $\quad \therefore y_0'' = y_0 = 0.2$

y''' = y' + y' + xy''

= 2y' + xy'' $\quad \therefore y_0''' = 2y_0' = 2$

y'''' = 2y'' + y'' + xy''' $\quad \therefore y_0'''' = 3y_0'' + xy_0''' = 0.6$

Putting these values in equation 1, we get

$$y = 0.2 + (0.4)1 + +\frac{(0.4)^2}{2!}0.2 + \frac{(0.4)^3}{3!}.2 + \frac{(0.4)^4}{4!}.(0.6) + \cdots$$

$$y = 0.2 + 0.4 + 0.016 + 0.02133 + 0.00064$$

$$\boxed{y = 0.63797.}$$

Q.2 a) Solve $\frac{d^2y}{dx^2} - 16y = x^2e^{3x} + e^{2x} - \cos 3x + 2^x$. [6]

ANS: The auxiliary equation is $D^2 - 16 = 0$

$\therefore D = 4, -4$

$\therefore$ The C.F. is $y = C_1e^{4x} + C_2e^{-4x}$

Now, to find P.I.,

P.I. = $\frac{1}{D^2-16}(x^2e^{3x} + e^{2x} - \cos 3x + 2^x)$

Now, $\frac{1}{D^2-16}x^2e^{3x} = e^{3x}.\frac{1}{(D+3)^2-16}.x^2$

$$= e^{3x}.\frac{1}{D^2+6D+9-16}.x^2 = e^{3x}.\frac{1}{D^2+6D-7}.x^2$$

$$= -\frac{e^{3x}}{7} \cdot \frac{1}{\left(1-\frac{D^2+6D}{7}\right)} \cdot x^2$$

$$= -\frac{e^{3x}}{7} \cdot \left(1 - \frac{D^2+6D}{7}\right)^{-1} \cdot x^2$$

$$= -\frac{e^{3x}}{7} \left(1 + \frac{D^2+6D}{7} + \frac{D^4+6D^3+36D^2}{49} + \cdots\right) \cdot x^2$$

$$= -\frac{e^{3x}}{7}\left(x^2 + \frac{12x+2}{7} + \frac{72}{49}\right) = -\frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right)$$

$\therefore \frac{1}{D^2-16} \cdot e^{2x} = e^{2x} \frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{12}$

$\therefore \frac{1}{D^2-16} \cdot \cos 3x = \frac{\cos 3x}{-9-16} = \frac{\cos 3x}{-25}$

$\therefore \frac{1}{D^2-16} \cdot 2^x = \frac{1}{D^2-16} \cdot e^{x \log 2} = \frac{e^{x \log 2}}{(\log 2)^2-16} \cdot = \frac{2^x}{(\log 2)^2-16}.$

$\therefore$P.I. = $-\frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}.$

$\therefore$ The complete equation is,

$$\boxed{y = C_1 e^{4x} + C_2 e^{-4x} - \frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}.}$$

b) Show that $\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} dx = \pi \sin^{-1} a \quad 0 \le a \le 1.$ **[6]**

ANS: Let I (a) be the given integral. By the rule of differentiation under the integral sign.

$$\frac{dI}{da} = \int_0^{\pi} \frac{df}{da}\, dx = \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{\cos x}{1+a\cos x}\, dx = \int_0^{\pi} \frac{dx}{1+a\cos x}$$

Put $t = tan\frac{x}{2}, dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

When x = 0, t = 0;

When x =π, t = $tan\frac{\pi}{2}$=∞

$$\therefore \frac{dI}{da} = \int_0^{\infty} \frac{1}{1+a.\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{2\,dt}{(1+t^2)+a(1-t^2)}.$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{2\,dt}{(1+a)+(1-a)t^2}.$$

$$\frac{dI}{da} = \frac{1}{1-a}\int_0^{\infty} \frac{2\,dt}{\left[\frac{1+a}{1-a}\right]+t^2}$$

$$\frac{dI}{da} = \frac{2}{1-a}\sqrt{\frac{1-a}{1+a}}.\left[\tan^{-1}\sqrt{\frac{1-a}{1+a}}\right]_0^{\infty}$$

$$\frac{dI}{da} = \frac{2}{\sqrt{1-a^2}}.\frac{\pi}{2}$$

$$\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}}.$$

Integrating both sides w.r.t. a, we get

$$I = \pi\ \sin^{-1} a + c$$

To find c, put a = 0

$$I(0) = \pi\ \sin^{-1} 0 + c, c = 0$$

$$\therefore I = \pi \sin^{-1} a$$

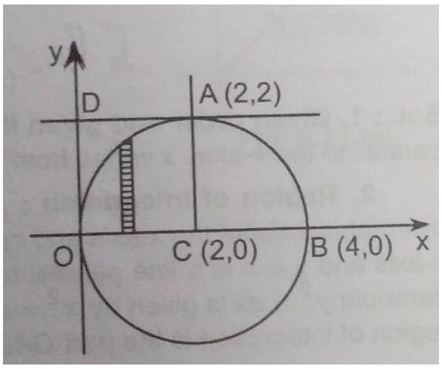
$$\boxed{\therefore \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\,dx = \pi \sin^{-1} a}$$

c) Change the order of integration and evaluate $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dxdy$. [8]

ANS: 1) Given order and given limits: Given order is: first w.r.t. x and then w.r.t y i.e., a strip parallel to the x-axis varies from x = $2-\sqrt{4-y^2}$ to x = $2-\sqrt{4-y^2}$. Y varies from y = 0 to y = 2.

2) Region of integration: x = $2-\sqrt{4-y^2}$ is the arc and x = $2+\sqrt{4-y^2}$ is the arc of the circle $(x-2)^2 + y^2 = 4$with centre at (2, 0) and radius = 2 above the x-axis. y = 0 is the x-axis and y = 2 is the line parallel to the x-axis through A (2, 2). The region of integration is the semi-circle OAB above the x-axis. The points of intersection of the circle and the x-axis are O (0, 0) and B (4, 0).

3) Change of order of integration: To change the order, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from y = 0 to y =$\sqrt{4-(x-2)^2}$ and then strip moves from x = 0 to x = 4.



$$I = \int_0^4 \int_0^{\sqrt{4-(x-2)^2}} \, dydx$$

$$I = \int_0^4 [y]_0^{\sqrt{4-(x-2)^2}} \, dx$$

$$I = \int_0^4 \sqrt{4-(x-2)^2} \, dx$$

$$I = \left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + 2\sin^{-1}\frac{x-2}{2}\right]_0^4$$

$$I = \left(2.\frac{\pi}{2}\right) - (-2.\frac{\pi}{2})$$

$$\boxed{\therefore \boldsymbol{I = 2\pi}}$$

Q.3 a) Evaluate $\iiint (x+y+z)\, dxdydz$ over the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x + y + z = 1. [6]

ANS:

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x+y+z) dzdydx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dydz$$

$$I = \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{1-x} [1-(x+y)^2] dydx$$

$$I = \frac{1}{2}\int_{x=0}^{1}\left[y - \frac{(x+y)^2}{2}\right]_0^{1-x} dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}[(1-x) - \frac{1}{3} + \frac{x^3}{3}]dx$$

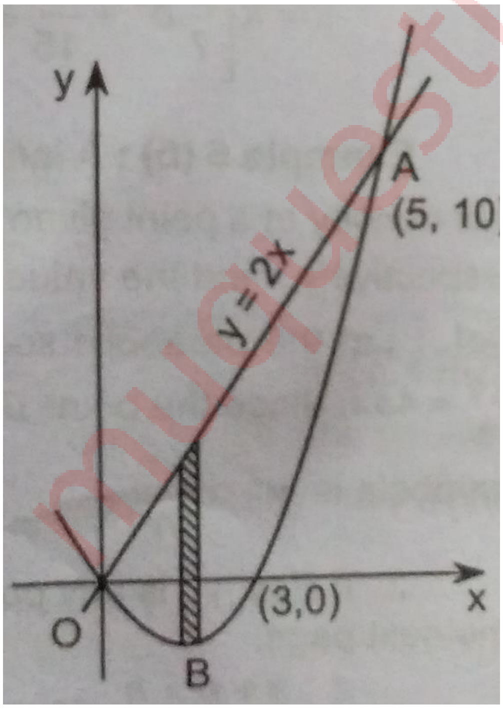
$$I = \frac{1}{2}\left[\frac{2x}{3} - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1$$

$$I = \frac{1}{2}.\frac{3}{12} = \frac{1}{8}$$

$$\therefore \boldsymbol{I = \frac{1}{8}}$$

b) Find the mass of lamina bounded by the curves $y = x^2 - 3x$ and $y = 2x$ if the density of the lamina at any point is given by $\frac{24}{25}xy$. [6]

ANS: The curve $y = x^2 - 3x$ i.e. $y + \frac{9}{4} = (x - \frac{3}{2})^2$ is parabola intersecting the x-axis in x = 0 and x = 3. The line y = 2x intersects this parabola at $x^2 - 3x = 2x$ i.e. $x^2 - 5x = 0$ i.e. at x = 0, x = 5. Therefore, points of intersection are (0,0) and (5,10). The surface density is ρ = (24/25)xy. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = x^2 - 3x$ to $y = 2x$ and then x varies from x = 0 to x = 5.



$\therefore$ Mass of lamina = $\int_0^5 \int_{x^2-3x}^{2x} \frac{24}{25} xydxdy$

$$= \frac{24}{25}\int_0^5 x\left[\frac{y^2}{2}\right]_{x^2-3x}^{2x} dx$$

$$= \frac{24}{50}\int_0^5 4x^3 - x(x^4 - 6x^3 + 9x^2)]dx$$

$$= \frac{24}{50}\int_0^5 -5x^3 + 6x^4 - x^5)]dx$$

$$= \frac{24}{50}\left[\frac{-x^6}{b} + \frac{6x^5}{5} - \frac{5x^4}{4}\right]_0^5$$

$$= \frac{24}{50}.5^4[-\frac{25}{6} + 6 - \frac{5}{4}]$$

$$= \frac{24}{50}.5^4.\frac{7}{12}$$

$\therefore$Mass of lamina = 175.

c) Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = \frac{\log x . \cos(\log x)}{x}$ **[8]**

ANS: Given that,

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = \frac{\log x . \cos(\log x)}{x}$$

Putting z = log x and x = e^z, we get

$$[D(D-1) + 3D + 3]y = e^{-z}.z.\cos z$$

$$[D^2 + 2D + 3]y = e^{-z}.z.\cos z$$

$\therefore$The A.E. is $D^2 + 2D + 3 = 0$

$$\therefore D = \frac{-2\pm 2\sqrt{2}.i}{2} = -1 \pm \sqrt{2}.i$$

$\therefore$The C.F. is $y = e^{-z}(C_1\cos\sqrt{2}z + C_2\sin\sqrt{2}z)$

P.I. $= \frac{1}{D^2+2D+3} e^{-z}.z.\cos z$

$$= e^{-z}.\frac{1}{(D-1)^2+2(D-1)+3}.z.\cos z = e^{-z}.\frac{1}{D^2+2}.z.\cos z$$

$$= e^{-z}\left[z - \frac{1}{D^2+2} . 2D\right] . \frac{1}{D^2+2} . \cos z$$

$$= e^{-z}\left[z - \frac{1}{D^2+2} . 2D\right] \cos z = e^{-z}\left[z\cos z + \frac{1}{D^2+2} . 2\sin z\right]$$

$$= e^{-z}[z\cos z + 2\sin z]$$

The complete solution is,

y = C.F. + P.I.

$$y = e^{-z}\left(C_1 \cos\sqrt{2}z + C_2 \sin\sqrt{2}z\right) + e^{-z}[z\cos z + 2\sin z]$$

$$\boxed{\boldsymbol{y = \frac{1}{x}\left(C_1 \cos\sqrt{2}\log x + C_2 \sin\sqrt{2}\log x\right) + \frac{1}{x}[\log x \cos\log x + 2\sin\log x]}}$$

Q.4 a) Find by double integration the area bounded by the parabola $\boldsymbol{y^2 = 4x}$ And $\boldsymbol{y = 2x - 4}$ [6]

ANS: The parabola $y^2 = 4x$ and the line $y = 2x - 4$ intersect where $(2x-4)^2 = 4x$

$\therefore 4x^2 - 16x + 16 = 4x$ $\quad$ $\therefore 4x^2 - 20x + 16 = 0$

$\therefore x^2 - 5x + 4 = 0$ $\quad$ $\therefore$ (x - 4) (x - 1)=0

$\therefore$ x = 1, 4.

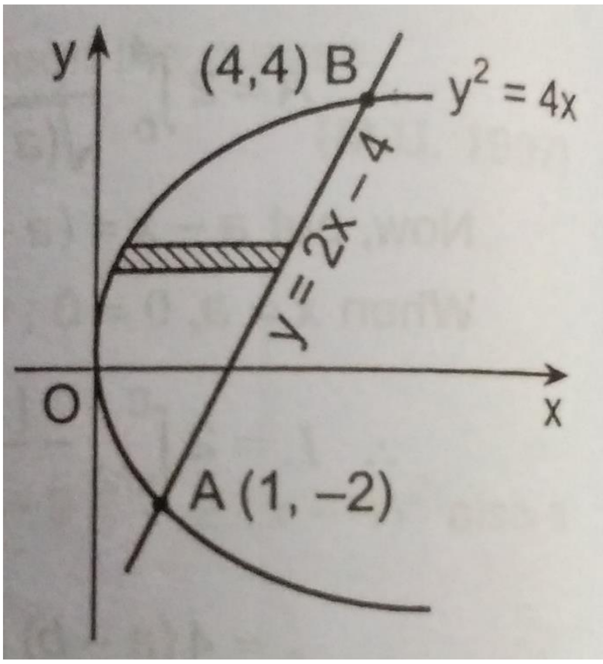
When x = 1, y = 2 – 4 = -2; and when x = 4, y = 8 - 4 = 4. Thus, the points of intersection are A (1, -2) and B (4, 4).

Now, consider a strip parallel to x-axis. On this strip x varies from x = $y^2/4$ to x = (y+4)/2. The strip then moves parallel to the x-axis from y = -2 to y = 4.

$$\therefore A = \int_{-2}^{4}\int_{y^2/4}^{(y+4)/2} dxdy = \int_{-2}^{4}[x]_{\frac{y^2}{4}}^{\frac{y+4}{2}} dy$$

$$= \int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right)dy$$

$$= \frac{1}{4}\int_{-2}^{4}(2y + 8 - y^2)dy$$



$$= \frac{1}{4}\left[y^2 + 8y - \frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}[\left(16 + 32 - \frac{64}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)]$$

$$= \frac{1}{4}(60 - 24)$$

$\therefore \boldsymbol{A = 9}$

b) Solve $\frac{dy}{dx} + x \sin 2y = x^3 cos^2 y$ [6]

ANS: Given, $\frac{dy}{dx} + x \sin 2y = x^3 cos^2 y$

Dividing both sides by $\cos^2 x$,

$$sec^2 x \frac{dy}{dx} + x\, sec^2 x \sin 2y = x^3$$

$$sec^2 x \frac{dy}{dx} + 2x \tan y = x^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Put tan y = v and differentiate w.r.t. x,

$$sec^2 x \frac{dy}{dx} = \frac{dv}{dx}$$

Hence, from (1), we get $\frac{dv}{dx} + 2v.x = x^3$

$\therefore\ P = 2x$ And Q = x^3

$\therefore \int P\, dx = \int 2x\, dx = x^2$

$\therefore$ I.F. = $e^{\int P\, dx} = e^{\int 2x\, dx} = e^{x^2}$

$\therefore$ The solution is $v\, e^{x^2} = \int e^{x^2} x^3\, dx + c$

To find the integral put $x^2 = t,\ xdx = \frac{dt}{2}$.

$\therefore I = \int e^t . t . \frac{dt}{2} = \frac{1}{2}[te^t - \int e^t . dt]$……….. [By parts]

$\therefore I = \frac{1}{2}[te^t - e^t] = \frac{1}{2}e^t(t-1) = \frac{1}{2}e^{x^2}(x^2-1)$

$\therefore$ The solution is $v\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$

$$\boxed{\therefore \mathbf{\tan} \boldsymbol{y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1)+c}}$$

c) Solve $\boldsymbol{\frac{dy}{dx} = x^3 + y}$ with initial conditions $\boldsymbol{y(0) = 2}$ at $\boldsymbol{x = 0.2}$ in step of

h = 0.1 by Runge Kutta method of Fourth order. [8]

ANS: Given that, $\frac{dy}{dx} = x^3 + y$

$f(x, y) = x^3 + y, x_0 = 0, y_0 = 2\ and\ h = 0.1$

$\therefore k_1 = hf(x_0, y_0) = 0.1(0+2) = 0.2$

$\therefore k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2}{2}] = 0.2100$

$\therefore k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.2105$

$\therefore k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.23105$

$\therefore k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{0.2 + 2(0.21) + 2(0.2105) + 0.23105}{6}$

$$\boxed{\therefore \boldsymbol{k = 0.2120}}$$

Q.5 a) Evaluate $\int_0^1 x^5 \sin^{-1} x\, dx$ and find the value of $\beta(\frac{9}{2},\frac{1}{2})$. **[6]**

ANS: $I = \int_0^1 x^5 \sin^{-1} x\, dx$

Put $\sin^{-1} x = t$ $\quad\therefore$ x = sin t $\quad dx = \cos t\, dt$

When x = 0, t = 0 $\quad$ when x = 1, t = $\pi/2$

$$I = \int_0^{\pi/2} \sin^5 t.t.\cos t\, dt = \int_0^{\pi/2} t(\sin^5 t.\cos t)\, dt$$

Integrating by parts,

$$I = \left[t.\frac{sin^6 x}{6}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{sin^6 x}{6}.1.dt$$

$$I = \left(\frac{\pi}{2}.\frac{1}{6} - 0\right) - \frac{1}{6}.\frac{5.3.1}{6.4.2}.\frac{\pi}{2}$$

$$I = \frac{\pi}{12} - \frac{5\pi}{192}$$

$$\boxed{\therefore \boldsymbol{I} = \frac{\mathbf{11\pi}}{\mathbf{192}}}$$

$$\beta\left(\frac{9}{2},\frac{1}{2}\right) = \frac{\left\lceil\frac{9}{2}\right\lceil\frac{1}{2}}{\lceil 5} = \frac{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\left\lceil\frac{1}{2}\right\lceil\frac{1}{2}}{5.4.3.2.1}$$

$$\boxed{\beta\left(\frac{9}{2},\frac{1}{2}\right) = \frac{105\pi}{384}}$$

b) In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt} + Ri = E$. Find the current i at time t at t = 0 and i = 0 and L, R and E are constants. **[6]**

ANS: The given equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is linear of the type $\frac{dy}{dx} + Py = Q$

$\therefore$ Its solution is $ie^{\int R/_L dt} = \int e^{\int R/_L dt}.\frac{E}{L}.dt + c$

$$i.e^{Rt/_L} = \frac{E}{L}\int e^{Rt/_L}\, dt + c = \frac{E}{L}.e^{Rt/_L}\frac{L}{R} + c$$

$$= \frac{E}{R}e^{Rt/_L} + c$$

When t = 0 and i = 0 $\therefore c = -\frac{E}{R}$

$\therefore i.e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R}$

$\therefore i = \frac{E}{R}(e^{Rt/L} - 1)$

$$\boxed{\therefore \boldsymbol{i = \frac{E}{R}(1 - e^{-Rt/L})}}$$

c) Evaluate $\int_0^6 \frac{dx}{1+3x}$ by using 1} Trapezoidal 2} Simpsons (1/3) rd. and 3} Simpsons (3/8) Th rule. [8]

ANS:

X	0	1	2	3	4	5	6
Y	1	0.25	0.1428	0.1	0.0769	0.0625	0.0526
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1} Trapezoidal Rule:

I = $\frac{h}{2}(X + 2R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

R = Sum of Remaining values = 0.25 + 0.1428 + 0.1 + 0.0769 + 0.0625 = 0.6322

I = $\frac{1}{2}\left(1.0526 + 2(0.6322)\right)$

$\boldsymbol{I = 1.1585}$

2} Simpsons (1/3) rd rule

$I = \frac{h}{3}(X + 2E + 4O)$

X = Sum of Extreme values =1 + 0.0526 = 1.0526

E = Sum of even ordinates = 0.1428 + 0.0769 = 0.2197

O = Sum of odd ordinates = 0.25 + 0.1 + 0.0625 = 0.4125

$I = \frac{1}{3}\left(1.0526 + 2(0.2197) + 4(0.4125)\right)$

$\boldsymbol{I = 0.5616}$.

3} Simpsons (3/8) Th rule.

$I = \frac{3h}{8}(X + 2T + 4R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

T = Sum of multiple of three =0.1

R = Sum of Remaining values =0.25 + 0.1428 +0.0769 + 0.0625 = 0.5322

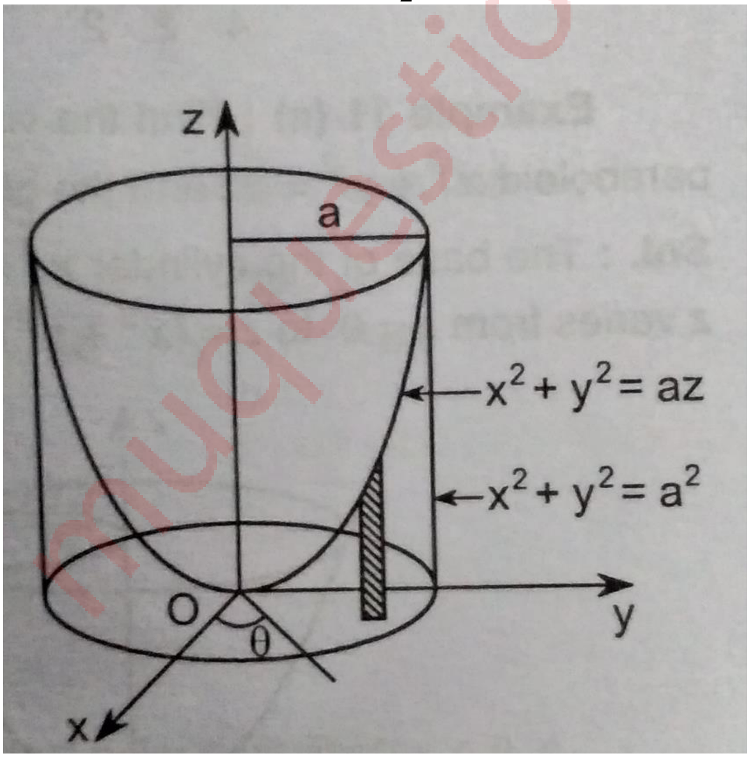
$I = \frac{3*1}{8}(1.0526 + 2(0.1) + 4(0.5322))$

$\boldsymbol{I = 1.06845}$.

Q.6 a) Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$. [6]

ANS: The equations of the cylinder and the paraboloid in polar form are r = a and r^2 = az.

Now, z varies from z = 0 to z = r^2/a, r varies from r = 0 to r = a and θ varies from θ = 0 to θ = $\frac{\pi}{2}$ taken 4 times.



$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\int_{z=0}^{r^2/a} r\,dr\,d\theta\,dz$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}} r\,[z]_0^{r^2/a} dr d\theta$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\frac{r^3}{a}dr\,d\theta$$

$$\therefore V = \frac{4}{a}\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^a d\theta$$

$$\therefore V = \frac{4}{a}\int_0^{\frac{\pi}{2}}\frac{a^4}{4}\,d\theta$$

$$\therefore V = a^3\int_0^{\frac{\pi}{2}} d\theta$$

$$\therefore V = a^3[\theta]_0^{\pi/2}$$

$$\boxed{\therefore \boldsymbol{V} = \frac{\boldsymbol{\pi a^3}}{\mathbf{2}}}$$

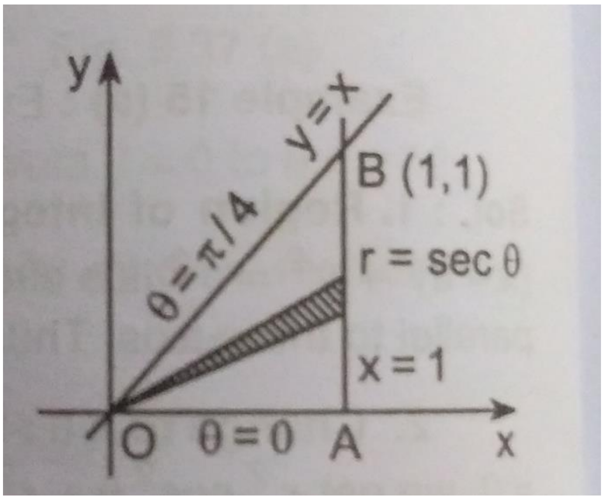
b) Change to polar coordinates and evaluate $\int_0^1\int_0^x (x+y)\,dydx$. [6]

ANS: 1) Region of integration: y= 0 is the x-axis and y = x is a line OB through the origin; x = 0 is the y-axis and x = 1 is a line AB parallel to the y-axis. Thus the region of integration is the triangle OAB.

2) Change to r, θ: Putting x = r cos θ and y = r sin θ, the line y = x becomes r sin θ= r cos θ i.e. tan θ = 1 i.e. θ = $\frac{\pi}{4}$. The x-axis is given by θ = 0 and the y-axis is given by θ =$\frac{\pi}{2}$. And line x = 1 is given by r cos θ = 1 i.e. r = sec θ.

3) Integrand: Putting x = r cos θ and y = r sin θ in (x + y), we get,

r cos θ + r sin θ = r(cos θ + sin θ) and dydx is replaced by r drdθ



$$\therefore I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \mathrm{r}(\cos\theta + \sin\theta)\mathrm{r}dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} (\cos\theta + \sin\theta)\mathrm{r}^2 dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^{\sec\theta} d\theta$$

$$I = \frac{1}{3}\int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\, sec^3\theta\, d\theta$$

$$I = \frac{1}{3}[\int_0^{\frac{\pi}{4}} sec^2\theta\, d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{cos^3\theta}\sin\theta\, d\theta]$$

$$I = \frac{1}{3}\left[\tan\theta + \frac{1}{2cos^2\theta}\right]_0^{\frac{\pi}{4}}$$

$$I = \frac{1}{3}(1 + \frac{1}{2}(2-1))$$

$$\boxed{\boldsymbol{I = \frac{1}{2}}}$$

c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}. \qquad \textbf{[8]}$$

ANS: A.E: $D^2 + 3D + 2 = 0$

Solving the equation, we get

$\therefore$ D = -1, -2.

$\therefore$ C.F = $C_1 e^{-x}$ + $C_2 e^{-2x}$.

$\therefore y_1 = e^{-x} \quad y_2 = e^{-2x}$

$\therefore y_1' = -e^{-x} \qquad y_2' = -2e^{-2x}$

$$\therefore w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x}$$

$$= -e^{-3x}$$

X $= e^{e^x}$.

$$\therefore u = -\int \frac{y_2 X}{w}\, dx$$

$$= -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}}\, dx$$

$$= -\int e^{e^x} . e^x\, dx$$

Put e^x = t

e^x dx =dt

$\therefore \int e^t dt = e^t + c$.

$\therefore w = e^{e^x} + c$

$$v = \int \frac{y_1 X}{w}\, dx$$

$$v = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}}\, dx$$

$$v = \int e^{e^x} e^{2x}\, dx$$

Putting e^x = t

$$\therefore v = \int e^t . t\, dt = te^t - e^t$$

$$\therefore v = e^x e^{e^x} - e^{e^x}$$

$\therefore$ P.I. = $uy_1 + vy_2 = e^{e^x}.e^{-x} - (e^x e^{e^x} - e^{e^x})\, e^{-2x}$

$= e^{-2x}.\, e^{e^x}$

$\therefore$ The complete solution is,

y = C.F + P.I

$$\boxed{\mathbf{y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x}.\, e^{e^x}}}$$

MUMBAI UNIVERSITY

SEMESTER – 2

APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1 a) Evaluate $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}dx$. **[3]**

ANS: $I = \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}dx.$

Put $x^3 = t$

$\therefore x = t^{\frac{1}{3}}$

$dx = \frac{1}{3}t^{\frac{-2}{3}}$

$\therefore I = \int_0^\infty e^{-t}.t^{-\frac{1}{6}}.\frac{1}{3}.t^{\frac{-2}{3}}\,dt$

$\therefore I = \int_0^\infty e^{-t}.t^{-\frac{5}{6}}\,dt$

$$\boxed{\therefore \boldsymbol{I} = \frac{1}{3}\Gamma\left(\frac{1}{6}\right).}$$

b) Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from y = 1 to y = 2.
[3]

ANS: We have $x = \frac{y^3}{3} + \frac{1}{4y}$

Diff w.r.t. y, we get

$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(y^2-\frac{1}{4y^2}\right)^2 = y^4+\frac{1}{2}+\frac{1}{16y^4} = \left(y^2+\frac{1}{4y^2}\right)^2$$

We know that,

$$s = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$s = \int_1^2 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\, dy$$

$$s = \left[\frac{y^3}{3}-\frac{1}{4y}\right]_1^2$$

$$s = \frac{8}{3}-\frac{1}{8}-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$\boxed{\boldsymbol{s = \frac{59}{24}}}$$

c) Solve $(\boldsymbol{D^2+D})\boldsymbol{y} = \boldsymbol{e^{4x}}$. **[3]**

ANS: For auxiliary equation,

$D^2+D = 0$

Solving we get,

D = -1, 0.

$\therefore$ C.F. = $C_1e^{-x}+C_2e^{0x}$

$\therefore C.F = C_1e^{-x}+C_2$

For P.I.,

$$y = \frac{e^{4x}}{D^2+D}$$

Now, put D = 4

$$\therefore y = \frac{e^{4x}}{4^2+4} = \frac{e^{4x}}{20}$$

$\therefore$ The complete solution is,

$$\boxed{\boldsymbol{y = C_1e^{-x}+C_2+\frac{e^{4x}}{20}}.}$$

d) Evaluate $\int_0^1\int_{x^2}^{x} \boldsymbol{xy(x+y)dydx}$. **[3]**

ANS: We have,

$$I = \int_0^1 \int_{x^2}^{x} xy(x + y)dydx.$$

$$I = \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$I = \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$I = \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$I = \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$\boxed{\boldsymbol{I = \frac{3}{56}}}$$

e) Solve $(4x + 3y - 4)dx + (3x - 7y - 3)dy = 0$. **[4]**

ANS: Given, $(4x + 3y - 4)dx + (3x - 7y - 3)dy = 0.$

$\therefore$ M = $(4x + 3y - 4)$ and N = $(3x - 7y - 3)$

Differentiating M by y and N by x, we get,

$\frac{dM}{dy} = 3$ And $\frac{dN}{dx} = 3$

$\therefore \frac{dM}{dy} = \frac{dN}{dx}$

$\therefore$ The given equations are exact.

For solution,

$\int M\, dx = \int (4x + 3y - 4)\, dx$

$\int M\, dx = 2x^2 + 3xy - 4x$

$\int (Term\ is\ N\ free\ from\ x) = \int -7y - 3\, dy$

$= \frac{-7y^2}{2} - 3y$

$\therefore$ The final solution is,

$$2x^2 + 3xy - 4x - \frac{7y^2}{2} - 3y = c$$

$$\boxed{4x^2 + 6xy - 8x - 7y^2 - 6y = c}$$

f) Solve $\frac{dy}{dx} = 1 + xy$ with initial condition $x_0 = 0,\ y_0 = 0.2$ By Taylors series method. Find the approximate value of y for x = 0.4(step size = 0.4).

ANS: The Taylor series is given by,

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0'''\ldots.. \qquad \ldots\ldots..\ (1)$$

With $x_0 = 0, y_0 = 0.2, x = 0.4$

Now, y' = 1 + xy $\quad \therefore y_0' = 1$

y'' = y + xy' $\quad \therefore y_0'' = y_0 = 0.2$

y''' = y' + y' + xy''

= 2y' + xy'' $\quad \therefore y_0''' = 2y_0' = 2$

y'''' = 2y'' + y'' + xy''' $\quad \therefore y_0'''' = 3y_0'' + xy_0''' = 0.6$

Putting these values in equation 1, we get

$$y = 0.2 + (0.4)1 + + \frac{(0.4)^2}{2!}0.2 + \frac{(0.4)^3}{3!}.2 + \frac{(0.4)^4}{4!}.(0.6) + \cdots$$

$$y = 0.2 + 0.4 + 0.016 + 0.02133 + 0.00064$$

$$\boxed{y = 0.63797.}$$

Q.2 a) Solve$\frac{d^2y}{dx^2} - 16y = x^2e^{3x} + e^{2x} - \cos 3x + 2^x$. [6]

ANS: The auxiliary equation is $D^2 - 16 = 0$

$\therefore D = 4, -4$

$\therefore$ The C.F. is $y = C_1e^{4x} + C_2e^{-4x}$

Now, to find P.I.,

P.I. = $\frac{1}{D^2-16}(x^2e^{3x} + e^{2x} - \cos 3x + 2^x)$

Now, $\frac{1}{D^2-16}x^2e^{3x} = e^{3x}.\frac{1}{(D+3)^2-16}.x^2$

$$= e^{3x}.\frac{1}{D^2+6D+9-16}.x^2 = e^{3x}.\frac{1}{D^2+6D-7}.x^2$$

$$= -\frac{e^{3x}}{7} \cdot \frac{1}{\left(1-\frac{D^2+6D}{7}\right)} \cdot x^2$$

$$= -\frac{e^{3x}}{7} \cdot \left(1 - \frac{D^2+6D}{7}\right)^{-1} \cdot x^2$$

$$= -\frac{e^{3x}}{7} (1 + \frac{D^2+6D}{7} + \frac{D^4+6D^3+36D^2}{49} + \cdots) \cdot x^2$$

$$= -\frac{e^{3x}}{7}\left(x^2 + \frac{12x+2}{7} + \frac{72}{49}\right) = -\frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right)$$

$\therefore \frac{1}{D^2-16} \cdot e^{2x} = e^{2x} \frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{12}$

$\therefore \frac{1}{D^2-16} \cdot \cos 3x = \frac{\cos 3x}{-9-16} = \frac{\cos 3x}{-25}$

$\therefore \frac{1}{D^2-16} \cdot 2^x = \frac{1}{D^2-16} \cdot e^{x\log 2} = \frac{e^{x\log 2}}{(\log 2)^2-16} \cdot = \frac{2^x}{(\log 2)^2-16}.$

$\therefore$P.I. = $-\frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}.$

$\therefore$ The complete equation is,

$$\boxed{\boldsymbol{y = C_1 e^{4x} + C_2 e^{-4x} - \frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}}.}$$

b) Show that $\boldsymbol{\int_0^\pi \frac{\log(1+a\cos x)}{\cos x} dx = \pi \sin^{-1} a \ \ 0 \le a \le 1}$. **[6]**

ANS: Let I (a) be the given integral. By the rule of differentiation under the integral sign.

$$\frac{dI}{da} = \int_0^\pi \frac{df}{da}\, dx = \int_0^\pi \frac{1}{\cos x} \cdot \frac{\cos x}{1+a\cos x}\, dx = \int_0^\pi \frac{dx}{1+a\cos x}$$

Put $t = tan\frac{x}{2}, dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

When x = 0, t = 0;

When x =π, t = $tan\frac{\pi}{2}$=∞

$$\therefore \frac{dI}{da} = \int_0^{\infty} \frac{1}{1+a.(\frac{1-t^2}{1+t^2})} \cdot \frac{2\,dt}{1+t^2}$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{2\,dt}{(1+t^2)+a(1-t^2)}.$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{2\,dt}{(1+a)+(1-a)t^2}.$$

$$\frac{dI}{da} = \frac{1}{1-a} \int_0^{\infty} \frac{2\,dt}{\left[\frac{1+a}{1-a}\right]+t^2}$$

$$\frac{dI}{da} = \frac{2}{1-a} \sqrt{\frac{1-a}{1+a}}. \left[\tan^{-1} \sqrt{\frac{1-a}{1+a}}\right]_0^{\infty}$$

$$\frac{dI}{da} = \frac{2}{\sqrt{1-a^2}}.\frac{\pi}{2}$$

$$\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}}.$$

Integrating both sides w.r.t. a, we get

$$I = \pi\ \sin^{-1} a + c$$

To find c, put a = 0

$$I(0) = \pi\ \sin^{-1} 0 + c, c = 0$$

$$\therefore I = \pi \sin^{-1} a$$

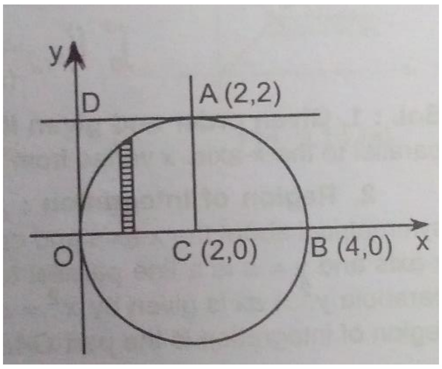
$$\boxed{\therefore \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} dx = \pi \sin^{-1} a}$$

c) Change the order of integration and evaluate $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dxdy$. [8]

ANS: 1) Given order and given limits: Given order is: first w.r.t. x and then w.r.t y i.e., a strip parallel to the x-axis varies from x = 2 $-\sqrt{4-y^2}$ to x = 2 $-\sqrt{4-y^2}$. Y varies from y = 0 to y = 2.

2) Region of integration: x = 2 $-\sqrt{4-y^2}$ is the arc and x = 2 + $\sqrt{4-y^2}$ is the arc of the circle $(x - 2)^2 + y^2 = 4$with centre at (2, 0) and radius = 2 above the x-axis. y = 0 is the x-axis and y = 2 is the line parallel to the x-axis through A (2, 2). The region of integration is the semi-circle OAB above the x-axis. The points of intersection of the circle and the x-axis are O (0, 0) and B (4, 0).

3) Change of order of integration: To change the order, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from y = 0 to y =$\sqrt{4-(x-2)^2}$ and then strip moves from x = 0 to x = 4.



$$I = \int_0^4 \int_0^{\sqrt{4-(x-2)^2}} dydx$$

$$I = \int_0^4 [y]_0^{\sqrt{4-(x-2)^2}} dx$$

$$I = \int_0^4 \sqrt{4-(x-2)^2}\, dx$$

$$I = \left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + 2\sin^{-1}\frac{x-2}{2}\right]_0^4$$

$$I = \left(2.\frac{\pi}{2}\right) - (-2.\frac{\pi}{2})$$

$$\therefore \boldsymbol{I = 2\pi}$$

Q.3 a) Evaluate $\iiint (x+y+z)\, dxdydz$ over the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x + y + z = 1. [6]

ANS:

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x+y+z) dzdydx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dydz$$

$$I = \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{1-x} [1-(x+y)^2] dydx$$

$$I = \frac{1}{2}\int_{x=0}^{1}\left[y - \frac{(x+y)^2}{2}\right]_0^{1-x} dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}[(1-x) - \frac{1}{3} + \frac{x^3}{3}]dx$$

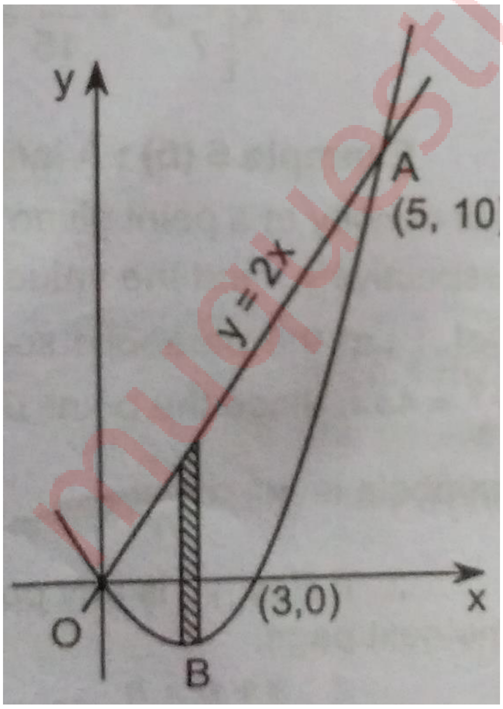
$$I = \frac{1}{2}\left[\frac{2x}{3} - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1$$

$$I = \frac{1}{2}.\frac{3}{12} = \frac{1}{8}$$

$$\therefore \boldsymbol{I = \frac{1}{8}}$$

b) Find the mass of lamina bounded by the curves $y = x^2 - 3x$ and $y = 2x$ if the density of the lamina at any point is given by $\frac{24}{25}xy$. [6]

ANS: The curve $y = x^2 - 3x$ i.e. $y + \frac{9}{4} = (x - \frac{3}{2})^2$ is parabola intersecting the x-axis in x = 0 and x = 3. The line y = 2x intersects this parabola at $x^2 - 3x = 2x$ i.e. $x^2 - 5x = 0$ i.e. at x = 0, x = 5. Therefore, points of intersection are (0,0) and (5,10). The surface density is ρ = (24/25)xy. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = x^2 - 3x$ to $y = 2x$ and then x varies from x = 0 to x = 5.



$\therefore$ Mass of lamina = $\int_0^5 \int_{x^2-3x}^{2x} \frac{24}{25} xydxdy$

$$= \frac{24}{25}\int_0^5 x\left[\frac{y^2}{2}\right]_{x^2-3x}^{2x} dx$$

$$= \frac{24}{50}\int_0^5 4x^3 - x(x^4 - 6x^3 + 9x^2)]dx$$

$$= \frac{24}{50}\int_0^5 -5x^3 + 6x^4 - x^5)]dx$$

$$= \frac{24}{50}\left[\frac{-x^6}{b} + \frac{6x^5}{5} - \frac{5x^4}{4}\right]_0^5$$

$$= \frac{24}{50}.5^4[-\frac{25}{6} + 6 - \frac{5}{4}]$$

$$= \frac{24}{50}.5^4.\frac{7}{12}$$

$\therefore$Mass of lamina = 175.

c) Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = \frac{\log x.\cos(\log x)}{x}$ **[8]**

ANS: Given that,

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = \frac{\log x.\cos(\log x)}{x}$$

Putting z = log x and x = e^z, we get

$$[D(D-1) + 3D + 3]y = e^{-z}.z.\cos z$$

$$[D^2 + 2D + 3]y = e^{-z}.z.\cos z$$

$\therefore$The A.E. is $D^2 + 2D + 3 = 0$

$$\therefore D = \frac{-2\pm2\sqrt{2}.i}{2} = -1 \pm \sqrt{2}.i$$

$\therefore$The C.F. is $y = e^{-z}(C_1\cos\sqrt{2}z + C_2\sin\sqrt{2}z)$

P.I. =$\frac{1}{D^2+2D+3}e^{-z}.z.\cos z$

$$= e^{-z}.\frac{1}{(D-1)^2+2(D-1)+3}.z.\cos z = e^{-z}.\frac{1}{D^2+2}.z.\cos z$$

$$= e^{-z}\left[z - \frac{1}{D^2+2}.2D\right].\frac{1}{D^2+2}.\cos z$$

$$= e^{-z}\left[z - \frac{1}{D^2+2}.2D\right]\cos z = e^{-z}\left[zcos\ z + \frac{1}{D^2+2}.2\sin z\right]$$

$$= e^{-z}[z\cos z + 2\sin z]$$

The complete solution is,

y = C.F. + P.I.

$$y = e^{-z}\left(C_1\cos\sqrt{2}z + C_2\sin\sqrt{2}z\right) + e^{-z}[z\cos z + 2\sin z]$$

$$\boldsymbol{y = \frac{1}{x}\left(C_1\cos\sqrt{2}\log x + C_2\sin\sqrt{2}\log x\right) + \frac{1}{x}[\log x\cos\log x + 2\sin\log x]}$$

Q.4 a) Find by double integration the area bounded by the parabola $\boldsymbol{y^2 = 4x}$ And $\boldsymbol{y = 2x - 4}$ [6]

ANS: The parabola $y^2 = 4x$ and the line $y = 2x - 4$ intersect where $(2x-4)^2 = 4x$

$\therefore 4x^2 - 16x + 16 = 4x$ $\therefore 4x^2 - 20x + 16 = 0$

$\therefore x^2 - 5x + 4 = 0$ $\therefore$ (x - 4) (x - 1)=0

$\therefore$x = 1, 4.

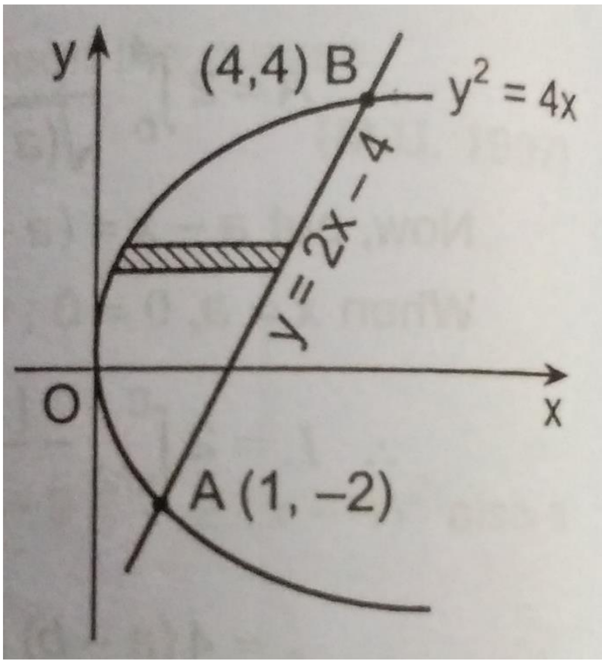
When x = 1, y = 2 – 4 = -2; and when x = 4, y = 8 - 4 = 4. Thus, the points of intersection are A (1, -2) and B (4, 4).

Now, consider a strip parallel to x-axis. On this strip x varies from x = $y^2/4$ to x = (y+4)/2. The strip then moves parallel to the x-axis from y = -2 to y = 4.

$$\therefore A = \int_{-2}^{4}\int_{y^2/4}^{(y+4)/2} dxdy = \int_{-2}^{4}[x]_{\frac{y^2}{4}}^{\frac{y+4}{2}} dy$$

$$= \int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right)dy$$

$$= \frac{1}{4}\int_{-2}^{4}(2y + 8 - y^2)dy$$



$$= \frac{1}{4}\left[y^2 + 8y - \frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}[\left(16 + 32 - \frac{64}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)]$$

$$= \frac{1}{4}(60 - 24)$$

$\therefore \boldsymbol{A = 9}$

b) Solve $\frac{dy}{dx} + x\sin 2y = x^3 cos^2 y$ [6]

ANS: Given, $\frac{dy}{dx} + x\sin 2y = x^3 cos^2 y$

Dividing both sides by $\cos^2 x$,

$$sec^2 x \frac{dy}{dx} + x\, sec^2 x \sin 2y = x^3$$

$$sec^2 x \frac{dy}{dx} + 2x \tan y \; = x^3 \text{……………………} (1)$$

Put tan y = v and differentiate w.r.t. x,

$$sec^2 x \frac{dy}{dx} = \frac{dv}{dx}$$

Hence, from (1), we get $\frac{dv}{dx} + 2v.x = x^3$

$\therefore\ P = 2x$ And Q = x^3

$\therefore \int P\,dx = \int 2x\,dx = x^2$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ The solution is $v\,e^{x^2} = \int e^{x^2} x^3\,dx + c$

To find the integral put $x^2 = t,\ xdx = \frac{dt}{2}$.

$\therefore I = \int e^t.t.\frac{dt}{2} = \frac{1}{2}[te^t - \int e^t.dt]$………. [By parts]

$\therefore I = \frac{1}{2}[te^t - e^t] = \frac{1}{2}e^t(t-1) = \frac{1}{2}e^{x^2}(x^2-1)$

$\therefore$ The solution is $v\,e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$

$$\boxed{\therefore \tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c}$$

c) Solve $\frac{dy}{dx} = x^3 + y$ with initial conditions $y(0) = 2$ at $x = 0.2$ in step of

h = 0.1 by Runge Kutta method of Fourth order. [8]

ANS: Given that, $\frac{dy}{dx} = x^3 + y$

$f(x,y) = x^3 + y, x_0 = 0, y_0 = 2\ and\ h = 0.1$

$\therefore k_1 = hf(x_0, y_0) = 0.1(0+2) = 0.2$

$\therefore k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2}{2}] = 0.2100$

$\therefore k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.2105$

$\therefore k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.23105$

$\therefore k = \frac{k_1+2k_2+2k_3+k_4}{6} = \frac{0.2+2(0.21)+2(0.2105)+0.23105}{6}$

$$\boxed{\therefore k = 0.2120}$$

Q.5 a) Evaluate $\int_0^1 x^5 \sin^{-1} x\, dx$ and find the value of $\beta(\frac{9}{2},\frac{1}{2})$. **[6]**

ANS: $I = \int_0^1 x^5 \sin^{-1} x\, dx$

Put $\sin^{-1} x = t$ $\therefore$ x = sin t $dx = \cos t\, dt$

When x = 0, t = 0 when x = 1, t = $\pi/2$

$$I = \int_0^{\pi/2} \sin^5 t.t.\cos t\, dt = \int_0^{\pi/2} t(\sin^5 t.\cos t)\, dt$$

Integrating by parts,

$$I = \left[t.\frac{sin^6x}{6}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{sin^6x}{6}.1.dt$$

$$I = \left(\frac{\pi}{2}.\frac{1}{6} - 0\right) - \frac{1}{6}.\frac{5.3.1}{6.4.2}.\frac{\pi}{2}$$

$$I = \frac{\pi}{12} - \frac{5\pi}{192}$$

$$\boxed{\therefore \boldsymbol{I} = \frac{\mathbf{11\pi}}{\mathbf{192}}}$$

$$\beta\left(\frac{9}{2},\frac{1}{2}\right) = \frac{\left\lceil\frac{9}{2}\right.\left\lceil\frac{1}{2}\right.}{\lceil 5} = \frac{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\left\lceil\frac{1}{2}\right.\left\lceil\frac{1}{2}\right.}{5.4.3.2.1}$$

$$\boxed{\beta\left(\frac{9}{2},\frac{1}{2}\right) = \frac{105\pi}{384}}$$

b) In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt} + Ri = E$.Find the current i at time t at t = 0 and i = 0 and L, R and E are constants. **[6]**

ANS: The given equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is linear of the type $\frac{dy}{dx} + Py = Q$

$\therefore$ Its solution is $ie^{\int R/_L dt} = \int e^{\int R/_L dt}.\frac{E}{L}.dt + c$

$$i.e^{Rt/_L} = \frac{E}{L}\int e^{Rt/_L}\, dt + c = \frac{E}{L}.e^{Rt/_L}\frac{L}{R} + c$$

$$= \frac{E}{R}e^{Rt/_L} + c$$

When t = 0 and i = 0 $\therefore c = -\frac{E}{R}$

$\therefore i.e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R}$

$\therefore i = \frac{E}{R}(e^{Rt/L} - 1)$

$\boxed{\therefore \boldsymbol{i} = \frac{\boldsymbol{E}}{\boldsymbol{R}}(\boldsymbol{1} - \boldsymbol{e}^{-\boldsymbol{Rt}/\boldsymbol{L}})}$

c) Evaluate $\int_0^6 \frac{dx}{1+3x}$ by using 1} Trapezoidal 2} Simpsons (1/3) rd. and 3} Simpsons (3/8) Th rule. [8]

ANS:

X	0	1	2	3	4	5	6
Y	1	0.25	0.1428	0.1	0.0769	0.0625	0.0526
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1} Trapezoidal Rule:

I = $\frac{h}{2}(X + 2R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

R = Sum of Remaining values = 0.25 + 0.1428 + 0.1 + 0.0769 + 0.0625 = 0.6322

I = $\frac{1}{2}\left(1.0526 + 2(0.6322)\right)$

$\boldsymbol{I = 1.1585}$

2} Simpsons (1/3) rd rule

$I = \frac{h}{3}(X + 2E + 4O)$

X = Sum of Extreme values =1 + 0.0526 = 1.0526

E = Sum of even ordinates = 0.1428 + 0.0769 = 0.2197

O = Sum of odd ordinates = 0.25 + 0.1 + 0.0625 = 0.4125

$I = \frac{1}{3}\left(1.0526 + 2(0.2197) + 4(0.4125)\right)$

$\boldsymbol{I = 0.5616}$.

3} Simpsons (3/8) Th rule.

$I = \frac{3h}{8}(X + 2T + 4R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

T = Sum of multiple of three =0.1

R = Sum of Remaining values =0.25 + 0.1428 +0.0769 + 0.0625 = 0.5322

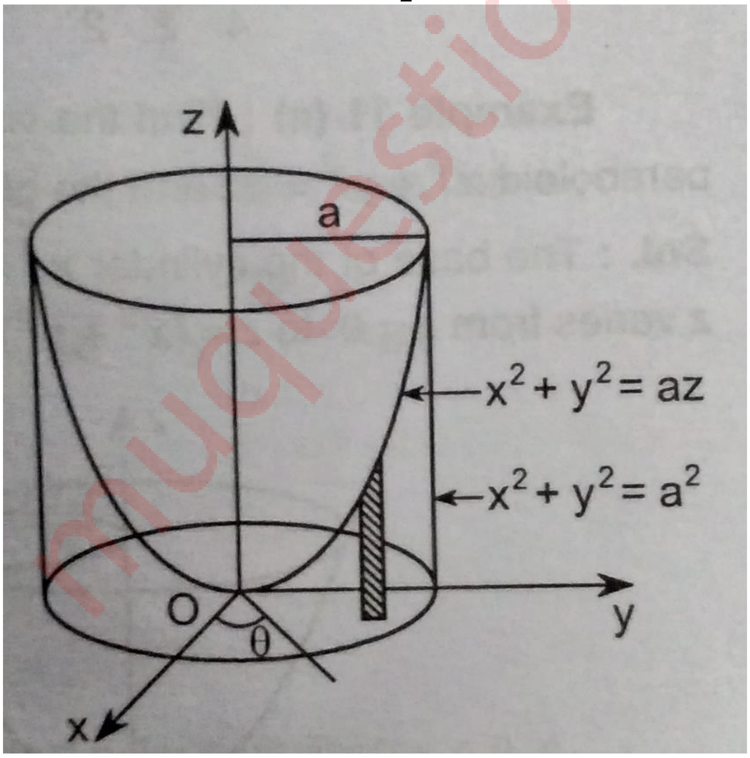
$I = \frac{3*1}{8}(1.0526 + 2(0.1) + 4(0.5322))$

$\boldsymbol{I = 1.06845}$.

Q.6 a) Find the volume bounded by the paraboloid $\boldsymbol{x^2 + y^2 = az}$ and the cylinder $\boldsymbol{x^2 + y^2 = a^2}$. [6]

ANS: The equations of the cylinder and the paraboloid in polar form are r = a and r^2 = az.

Now, z varies from z = 0 to z = r^2/a, r varies from r = 0 to r = a and θ varies from θ = 0 to θ = $\frac{\pi}{2}$ taken 4 times.



$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\int_{z=0}^{r^2/a} r\,dr\,d\theta\,dz$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}} r\,[z]_0^{r^2/a} dr d\theta$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\frac{r^3}{a}dr\,d\theta$$

$$\therefore V = \frac{4}{a}\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^a d\theta$$

$$\therefore V = \frac{4}{a}\int_0^{\frac{\pi}{2}}\frac{a^4}{4}\,d\theta$$

$$\therefore V = a^3\int_0^{\frac{\pi}{2}} d\theta$$

$$\therefore V = a^3[\theta]_0^{\pi/2}$$

$$\boxed{\therefore \boldsymbol{V} = \frac{\boldsymbol{\pi a^3}}{\boldsymbol{2}}}$$

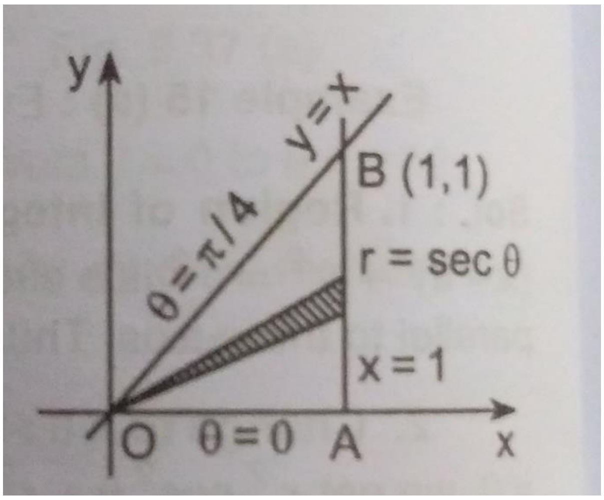
b) Change to polar coordinates and evaluate $\int_0^1\int_0^x (x+y)\,dydx$. **[6]**

ANS: 1) Region of integration: y= 0 is the x-axis and y = x is a line OB through the origin; x = 0 is the y-axis and x = 1 is a line AB parallel to the y-axis. Thus the region of integration is the triangle OAB.

2) Change to r, θ: Putting x = r cos θ and y = r sin θ, the line y = x becomes r sin θ= r cos θ i.e. tan θ = 1 i.e. θ = $\frac{\pi}{4}$. The x-axis is given by θ = 0 and the y-axis is given by θ =$\frac{\pi}{2}$. And line x = 1 is given by r cos θ = 1 i.e. r = sec θ.

3) Integrand: Putting x = r cos θ and y = r sin θ in (x + y), we get,

r cos θ + r sin θ = r(cos θ + sin θ) and dydx is replaced by r drdθ



$$\therefore I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \mathrm{r}(\cos\theta + \sin\theta)\mathrm{r}dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} (\cos\theta + \sin\theta)\mathrm{r}^2 dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^{\sec\theta} d\theta$$

$$I = \frac{1}{3}\int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\, sec^3\theta\, d\theta$$

$$I = \frac{1}{3}[\int_0^{\frac{\pi}{4}} sec^2\theta\, d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{cos^3\theta} \sin\theta\, d\theta]$$

$$I = \frac{1}{3}\left[\tan\theta + \frac{1}{2cos^2\theta}\right]_0^{\frac{\pi}{4}}$$

$$I = \frac{1}{3}(1 + \frac{1}{2}(2\text{-}1))$$

$$\boxed{\boldsymbol{I = \frac{1}{2}}}$$

c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}. \qquad \textbf{[8]}$$

ANS: A.E: $D^2 + 3D + 2 = 0$

Solving the equation, we get

$\therefore$ D = -1, -2.

$\therefore$ C.F = $C_1 e^{-x} + C_2 e^{-2x}$.

$\therefore y_1 = e^{-x} \quad y_2 = e^{-2x}$

$\therefore y_1' = -e^{-x} \qquad y_2' = -2e^{-2x}$

$$\therefore w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x}$$

$$= -e^{-3x}$$

X $= e^{e^x}$.

$$\therefore u = -\int \frac{y_2 X}{w}\, dx$$

$$= -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}}\, dx$$

$$= -\int e^{e^x} . e^x\, dx$$

Put e^x = t

e^x dx =dt

$\therefore \int e^t dt = e^t + c$.

$\therefore w = e^{e^x} + c$

$$v = \int \frac{y_1 X}{w}\, dx$$

$$v = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}}\, dx$$

$$v = \int e^{e^x} e^{2x}\, dx$$

Putting e^x = t

$$\therefore v = \int e^t . t\, dt = te^t - e^t$$

$$\therefore v = e^x e^{e^x} - e^{e^x}$$

$\therefore$ P.I. = $uy_1 + vy_2 = e^{e^x}.e^{-x} - (e^x e^{e^x} - e^{e^x})\, e^{-2x}$

$= e^{-2x}.\, e^{e^x}$

$\therefore$ The complete solution is,

y = C.F + P.I

$$\boxed{\mathbf{y = C_1e^{-x} + C_2e^{-2x} + e^{-2x}.\, e^{e^x}}}$$

MUMBAI UNIVERSITY

SEMESTER – II

APPLIED MATHEMATICS - II

QUESTION PAPER – MAY 2019

Q.1

a) Evaluate $\int_0^\infty y^4 e^{-y^6} dy$

Solution :

Let I = $\int_0^\infty y^4 e^{-y^6} dy$ and $y^6 = t$

$$y = t^{\frac{1}{6}}$$

$$dy = \frac{dt}{6t^{\frac{5}{6}}}$$

When y=0, t=0 and when y=∞, t=∞

Now,

I = $\int_0^\infty y^4 e^{-y^6} dy$

= $\int_0^\infty \left(t^{\frac{1}{6}}\right)^4 e^{-t} \frac{dt}{6y^{\frac{5}{6}}}$

= $\int_0^\infty t^{\frac{-1}{6}} e^{-t} dt$

= $\Gamma\left(\frac{5}{6}\right)$

$$\boxed{\int_0^\infty y^4 e^{-y^6} dy = \Gamma\left(\frac{5}{6}\right)}$$

b) Find the circumference of a circle of radius r by using parametric equations of the circle x=rcosɵ, y= rsinɵ.

Solution :

For a circle with radius r and parametric equations x=rcosɵ and y= rsinɵ,

Circumference, c = $\int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

= $\int_0^{2\pi} \sqrt{(-rsin\theta)^2 + (rcos\theta)^2} d\theta$

= $\int_0^{2\pi} r\sqrt{sin^2\theta + cos^2\theta} d\theta$

= r$\int_0^{2\pi} d\theta$

= r$[\theta]_0^{2\pi}$

$\boxed{c = 2\pi r}$

c) Solve $(D^2 + D - 6)y = e^{4x}$

Solution :

The auxiliary equation is $D^2 + D - 6 = 0$.

$(D-2)(D+3) = 0$

$D = 2, -3$

Complementary Function, $\boxed{C.F. = c_1e^{2x} + c_2e^{-3x}}$

Particular Integral, $P.I. = \frac{1}{(D-2)(D+3)}e^{4x}$

$= \frac{1}{(4-2)(4+3)}e^{4x}$

$= \frac{1}{2 \text{ x } 7}e^{4x}$

$$\boxed{P.I. = \frac{e^{4x}}{14}}$$

The complete solution is y = C.F. + P.I.

$$\boxed{y = c_1e^{2x} + c_2e^{-3x} + \frac{e^{4x}}{14}}$$

d) Evaluate $\int_0^1 \int_{x^2}^{x} xy(x^2 + y^2)dydx$

Solution :

Let I = $\int_0^1 \int_{x^2}^{x} xy(x^2 + y^2)dydx$

I = $\int_0^1 \int_{x^2}^{x} x^3y + y^3xdydx$

Integrating w.r.t y,

I = $\int_0^1 \left[x^3\frac{y^2}{2} + \frac{y^4}{4}x\right]_{x^2}^{x} dx$

I = $\int_0^1 x^3\frac{x^2}{2} + \frac{x^4}{4}x - x^3\frac{(x^2)^2}{2} - \frac{(x^2)^4}{4}x\ dx$

I = $\int_0^1 \frac{x^5}{2} + \frac{x^5}{4} - \frac{x^7}{2} - \frac{x^9}{4}\ dx$

I = $\int_0^1 \frac{3x^5}{4} - \frac{x^7}{2} - \frac{x^9}{4}\ dx$

Integrating w.r.t x,

I = $\left[\frac{3x^6}{4\ X\ 6} - \frac{x^8}{2\ X\ 8} - \frac{x^{10}}{4\ X\ 10}\right]_0^1$

I = $\frac{3}{24} - \frac{1}{16} - \frac{1}{40}$

I = $\frac{3}{80}$

$$\boxed{\int_0^1 \int_{x^2}^{x} xy(x^2 + y^2)dydx = \frac{3}{80}}$$

e) Solve (tany + x)dx + (xsec^{2}y – 3y)dy =0

Solution :

Comparing the equation (tany + x)dx + (xsec^{2}y – 3y)dy =0 with Mdx + Ndy = 0,

M = tany + x

N = xsec^{2}y – 3y

$$\frac{\partial M}{\partial y} = sec^2 y \qquad \frac{\partial N}{\partial x} = sec^2 y$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given D.E. is exact

$$\int Mdx = \int (tany + x)\, dx = xtany + \frac{x^2}{2}$$

$$\int (Terms\ in\ N\ free\ from\ x)dy = \int -3ydy = \frac{-3y^2}{2}$$

Solution,

$$\int Mdx + \int (Terms\ in\ N\ free\ from\ x)dy = c$$

$$\boxed{xtany + \frac{x^2}{2} - \frac{3y^2}{2} = c}$$

f) Solve $\frac{dy}{dx} = 1 + xy$ with initial condition $x_0 = 0$, $y_0 = 0.2$ by Euler's method. Find the approximate value of y at x = 0.4 with h = 0.1

Solution :

Since $f(x,y) = 1 + xy$, $f(x_0,y_0) = 1 + (0 \times 0.2) = 1$

At $x_1 = 0.1$, $y_1 = y_0 + h\, f(x_0,y_0) = 0.2 + \{0.1 \times [1 + (0 \times 0.2)]\} = 0.2 + 0.1 = 0.3$

At $x_2 = 0.2$, $y_2 = y_1 + h\, f(x_1,y_1) = 0.3 + \{0.1 \times [1 + (0.1 \times 0.3)]\} = 0.3 + 0.103 = 0.403$

At $x_3 = 0.3$, $y_3 = y_2 + h\, f(x_2,y_2) = 0.403 + \{0.1 \times [1 + (0 \times 0.2)]\} = 0.2 + 0.1 = 0.511$

At $x_4 = 0.4$, $y_4 = y_3 + h\, f(x_3,y_3) = 0.2 + \{0.1 \times [1 + (0 \times 0.2)]\} = 0.2 + 0.1 = 0.6263$

$\boxed{\text{At } x = 0.4,\ y = 0.6263}$

Q.2
a) Solve $(D^2 - 4D + 3)y = e^x\cos 2x + x^2$

Solution :

The auxiliary equation is $D^2 - 4D + 3$

$(D-3)(D-1) = 0$

$D = 3, 1$

Complementary Function, $\boxed{\text{C.F.} = c_1e^{3x} + c_2e^x}$

$$\text{Particular Integral , P.I.} = \frac{1}{(D-3)(D-1)}(e^x\cos 2x + x^2)$$

$$= \frac{1}{(D-3)(D-1)}e^x\cos 2x + \frac{1}{(D-3)(D-1)}x^2 \text{Type equation here.}$$

$$= e^x\frac{1}{(D+1-3)(D+1-1)}\cos 2x + 3\left(1-\frac{D}{3}\right)^{-1}(1-D)^{-1}x^2$$

$$= e^x\frac{1}{(D-2)(D)}\cos 2x + 3\left(1+\frac{D}{3}+\frac{D^2}{9}\right)(1+D+D^2)x^2$$

$$= e^x\frac{1}{D^2-2D}\cos 2x + 3\left(1+\frac{D}{3}+\frac{D^2}{9}\right)(x^2+2x+2)$$

$$= e^x\frac{1}{-4-2D}\cos 2x + 3\left(x^2+2x+2+\frac{2x}{3}+\frac{2}{3}+\frac{2}{9}\right)$$

$$= -\frac{e^x}{2}\frac{1}{D+2}\cos 2x + 3x^2 + 8x + \frac{26}{3}$$

$$= -\frac{e^x}{2}\frac{D-2}{D^2-4}\cos 2x + 3x^2 + 8x + \frac{26}{3}$$

$$= -\frac{e^x}{2}\frac{D-2}{-4-4}\cos 2x + 3x^2 + 8x + \frac{26}{3}$$

$$= \frac{e^x}{16}(-2\sin 2x - 2\cos 2x) + 3x^2 + 8x + \frac{26}{3}$$

$$= \frac{-e^x}{8}(\sin 2x + \cos 2x) + 3x^2 + 8x + \frac{26}{3}$$

$$= \frac{-e^x}{8}\sqrt{2}\cos\left(2x-\frac{\pi}{4}\right) + 3x^2 + 8x + \frac{26}{3}$$

The complete solution is y = C.F. + P.I.

$$\boxed{y = c_1e^{3x} + c_2e^x - \frac{e^x}{8}\sqrt{2}\cos\left(2x-\frac{\pi}{4}\right) + 3x^2 + 8x + \frac{26}{3}}$$

b) Show that $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} log(1+a)$

Solution :

$$I(a) = \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx$$

By the rule of differentiation under integral sign we have, differentiating w.r.t a,

$$\frac{dI}{da} = \int_0^{\infty} \frac{\partial}{\partial a}\left(\frac{\tan^{-1} ax}{x(1+x^2)}\right) dx$$

$$= \int_0^{\infty} \left(\frac{x}{1+a^2x^2} \cdot \frac{1}{x(1+x^2)}\right) dx$$

$$= \int_0^{\infty} \left(\frac{1}{(1+a^2x^2)} \cdot \frac{1}{(1+x^2)}\right) dx$$

$$= \frac{1}{1-a^2} \int_0^{\infty} \left(\frac{1}{(1+x^2)} - \frac{a^2}{(1+a^2x^2)}\right) dx$$

$$= \frac{1}{1-a^2} [\tan^{-1} x - a \tan^{-1} ax]_0^{\infty}$$

$$= \frac{1}{1-a^2} \left(\frac{\pi}{2} - a\frac{\pi}{2}\right)$$

$$\frac{dI}{da} = \frac{\pi}{2} \cdot \frac{1}{1+a}$$

Integrating both sides w.r.t a,

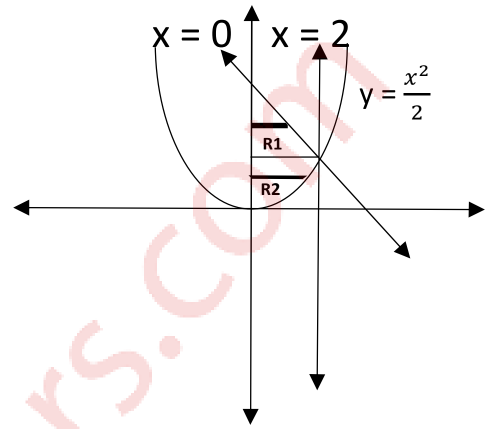
$$I = \int \frac{\pi}{2} \cdot \frac{1}{1+a} \, da$$

$$I = \frac{\pi}{2} \log(1+a)$$

$$\boxed{\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} log(1+a)}$$

c) Change the order of integration and evaluate $\int_0^2 \int_{\frac{x^2}{2}}^{4-x} xydydx$

Solution : Let I $= \int_0^2 \int_{\frac{x^2}{2}}^{4-x} xydydx$



x = 2, x = 0, y = 4-x, $y = \frac{x^2}{2}$

After changing the order of integration, we get two parts, R1 and R2 of the common region where the limits of the variables do not change.

In R1, x varies from 0 to 4-y and varies from 2 to 4

In R2, x varies from 0 to $\sqrt{2y}$ and varies from 0 to 2

$$I = \int_2^4 \int_0^{4-y} xydxdy + \int_0^2 \int_0^{\sqrt{2y}} xydxdy$$

$$I = \int_2^4 y \left[\frac{x^2}{2}\right]_0^{4-y} dy + \int_0^2 y \left[\frac{x^2}{2}\right]_0^{\sqrt{2y}} dy$$

$$I = \int_2^4 y \frac{(4-y)^2}{2} dy + \int_0^2 y \frac{(\sqrt{2y})^2}{2} dy$$

$$I = \int_2^4 y \frac{(16-8y+y^2)}{2} dy + \int_0^2 y \frac{2y}{2} dy$$

$$I = \frac{1}{2}\int_2^4 (16y - 8y^2 + y^3)\, dy + \int_0^2 y^2\, dy$$

$$I = \frac{1}{2}\left[8y^2 - \frac{8y^3}{3} + \frac{y^4}{4}\right]_2^4 + \left[\frac{y^3}{3}\right]_0^2$$

$$I = \frac{1}{2}\left[8.4^2 - \frac{8.4^3}{3} + \frac{4^4}{4} - 8.2^2 + \frac{8.2^3}{3} - \frac{2^4}{4}\right] + \left[\frac{2^3}{3}\right]$$

$$I = \frac{10}{3} + \frac{8}{3}$$

I = 6

$$\boxed{\int_0^2 \int_{\frac{x^2}{2}}^{4-x} xydydx = 6}$$

Q.3

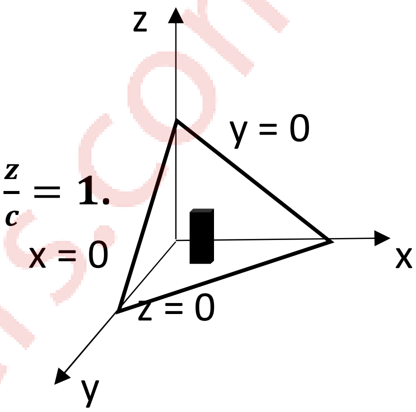
a) Evaluate $\iiint x^2yzdxdydz$ throughout the volume bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

<u>**Solution**</u> :

Let x = au, y = bv, z = cw

dx = a.du, dy = b.dv, dz = c.dw

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$



I = $\iiint x^2yzdxdydz$

I = $\iiint (au)^2.bv.cw.a.du.b.dv.c.dw$

I = $a^3b^2c^2 \iiint u^2vwdudv.dw$

The planes will become, u = 0, v = 0, w = 0 and u + v + w = 0.

If we consider an elementary cuboid, on this cuboid,

w varies from 0 to 1 – u – v

v varies from 0 to 1 – u

u varies from 0 to 1

I = $a^3b^2c^2 \int_0^1 \int_0^{1-u} \int_0^{1-u-v} u^2vwdwdvdu$

I = $a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2v \left[\frac{w^2}{2}\right]_0^{1-u-v} dvdu$

I = $a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2v \frac{(1-u-v)^2}{2} dvdu$

I = $a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2v \frac{[(1-u)^2-2(1-u)v+v^2]}{2} dvdu$

I = $a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2 \frac{[(1-u)^2v-2(1-u)v^2+v^3]}{2} dvdu$

I = $\frac{a^3b^2c^2}{2} \int_0^1 u^2[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}]_0^{1-u} du$

I = $\frac{a^3b^2c^2}{2} \int_0^1 u^2[\frac{(1-u)^4}{2} - \frac{2(1-u)^4}{3} + \frac{(1-u)^4}{4}]du$

I = $\frac{a^3b^2c^2}{2} \int_0^1 \frac{u^2(1-u)^4}{12} du$

I = $\frac{a^3b^2c^2}{24}$ β(3,5)

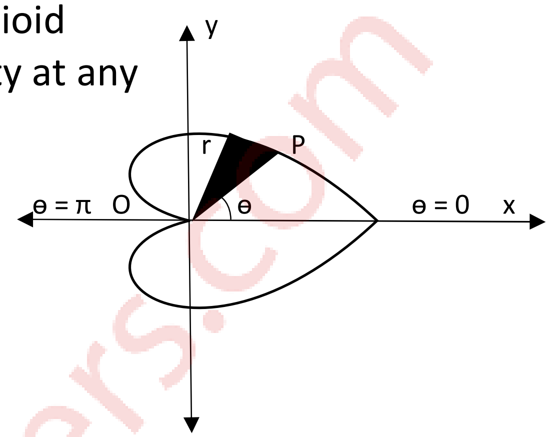
I = $\frac{a^3b^2c^2}{24}$ x $\frac{2!4!}{7!}$

I = $\frac{a^3b^2c^2}{2520}$

$\iiint x^2yzdxdydz = \frac{a^3b^2c^2}{2520}$ throughout the volume bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

b) Find the mass of the lamina of a cardioid r = a(1 + cosθ). If the density at any point varies as the square of its distance from its axis of symmetry.

Solution : Let P(r,θ) be any point on the given cardioid
The distance of P from the axis is r sinθ. The density at any point P(r,θ) is ρ = k $r^2 sin^2\theta$.
Consider a radial strip in the first quadrant.
On this strip, r varies from 0 to a(1 + cosθ) and θ varies from 0 to π.



Mass of the lamina,

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} (k\, r^2 sin^2\theta) r dr d\theta$$

$$= 2k\int_0^{\pi} sin^2\theta \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{ka^4}{2}\int_0^{\pi} sin^2\theta\,(1+cos\theta)^4 d\theta$$

$$= \frac{ka^4}{2}\int_0^{\pi}\left(2sin\frac{\theta}{2}cos\frac{\theta}{2}\right)^2\left(2cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 32\,ka^4\int_0^{\pi} sin^2\frac{\theta}{2}cos^{10}\frac{\theta}{2}d\theta$$

$$= 64\,ka^4\int_0^{\pi} sin^2 t\, cos^{10} t dt \qquad \left[\frac{\theta}{2} = t\right]$$

$$= 64\,ka^4\,\frac{1.9.7.5.3.1}{12.10.8.6.4.2.} \times \frac{\pi}{2}$$

$$= \frac{21}{32}ka^4\pi$$

$$\boxed{\text{Mass of the lamina } = \frac{21}{32}ka^4\pi}$$

c) Solve $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y= x^2+x+1$

<u>Solution</u> :

Let 3x +2 = v $\quad\frac{dv}{dx}=3$

$\frac{dy}{dx}=\frac{dy}{dv}x\frac{dv}{dx}$ = 3$\frac{dy}{dv}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(3\frac{dy}{dv}\right)=3\frac{d}{dv}\left(\frac{dy}{dv}\right)\frac{dv}{dx}=9\frac{d^2y}{dv^2}$$

The given equation changes to,

$$9v^2\frac{d^2y}{dv^2}+15v\frac{dy}{dv}-3y=\left(\frac{v-2}{3}\right)^2+\frac{v-2}{3}+1=\frac{v^2-4v+4}{9}+\frac{v-2}{3}+1$$

Multiplying throughout by 9,

$$81v^2\frac{d^2y}{dv^2}+135v\frac{dy}{dv}-27y=v^2-4v+4+3v-6+9$$

$$81v^2\frac{d^2y}{dv^2}+135v\frac{dy}{dv}-27y=v^2-v+7 \qquad \text{........(1)}$$

Put z = logv $\qquad$ v = e^z

Now,$v\frac{dy}{dv}=Dy,\ v^2\frac{d^2y}{dv^2}=D(D-1)y$

Equation (1) becomes,

[81D(D – 1) + 135D – 27]y = $e^{2z}-e^z+7$

[81$D^2+54D-27$]y = $e^{2z}-e^z+7$

The auxiliary equation is 81$D^2+54D-27$= 0.

(D + 1)(D - $\frac{1}{3}$) = 0

D = -1, $\frac{1}{3}$

Complementary Function, C.F. = $c_1e^{-z}+c_2e^{-z/3}$

= $c_1e^{-\log v}+c_2{}^{-\log v/3}$

= $c_1v^{-1}+c_2v^{-1/3}$

= $c_1(3x+2)^{-1}+c_2(3x+2)^{-1/3}$

Particular Integral, P.I. = $\frac{1}{81\quad{}^{2}+54D-27}e^{2z}-e^z+7$

$$= \frac{1}{81(2)^2+54(2)-27} e^{2z} - \frac{1}{81(1)^2+54(1)-27} e^{z} + \frac{1}{81(0)^2+54(0)-27} 7$$

$$= \frac{e^{2z}}{405} - \frac{e^{z}}{108} + \frac{7}{27}$$

$$= \frac{1}{27}\left(\frac{e^{2z}}{15} - \frac{e^{z}}{4} + 7\right)$$

Resubstituing z = log v

$$= \frac{1}{27}\left(\frac{e^{2logv}}{15} - \frac{e^{logv}}{4} + 7\right)$$

$$= \frac{1}{27}\left(\frac{v^2}{15} - \frac{v}{4} + 7\right)$$

Resubstituing v = 3x + 2

$$\text{P.I.} = \frac{1}{27}\left(\frac{(3x+2)^2}{15} - \frac{(3x+2)}{4} + 7\right)$$

The solution is,

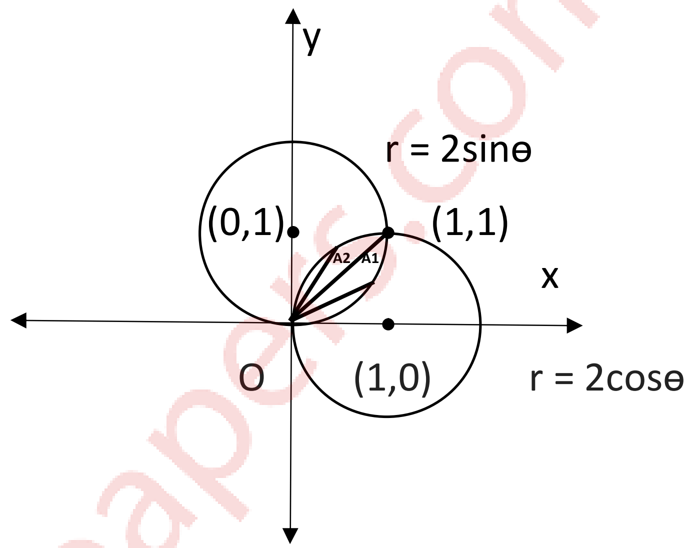
y = C.F. + P.I.

$$\boxed{y = c_1(3x+2)^{-1} + c_2(3x+2)^{-1/3} + \frac{1}{27}\left(\frac{(3x+2)^2}{15} - \frac{(3x+2)}{4} + 7\right)}$$

Q.4
a) Find by double integration the area common to the circles r = 2cosɵ and r = 2sinɵ.

Solution :

We have r = 2cosɵ



i.e. $\sqrt{x^2+y^2} = 2\frac{x}{\sqrt{x^2+y^2}}$

$x^2 + y^2 - 2x = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

Centre ≡ (1,0)

Radius = 1

Similarly, r = 2sinɵ

i.e. $\sqrt{x^2+y^2} = 2\frac{y}{\sqrt{x^2+y^2}}$

$x^2 + y^2 - 2y = 0$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$

Centre ≡ (0,1)

Radius = 1

Consider radial strips in both A1 and A2.

In A1, r varies from 0 to 2cosɵ and ɵ varies from 0 to π/4

In A2, r varies from 0 to 2sinɵ and ɵ varies from π/4 to π/2

Area = A1 + A2

$= \int_0^{\pi/4}\int_0^{2cos} rdrd\theta + \int_{\pi/4}^{\pi/2}\int_0^{2sin\theta} rdrd\theta$

$= \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{2cos\theta} d\theta + \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2sin\theta} d\theta$

$= 2\left[\int_0^{\frac{\pi}{4}}(cos^2\theta)\, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sin^2\theta d\theta\right]$

$= 2\int_0^{\frac{\pi}{4}}\frac{cos2\theta}{2} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1-cos2\theta}{2} d\theta$

$= [\frac{-sin2\theta}{2} + \theta]_0^{\frac{\pi}{4}} + [\theta + \frac{sin2\theta}{2}]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$$= \left(\frac{-\sin\frac{\pi}{2}}{2} + \frac{\pi}{4}\right) + \left(\frac{\pi}{2} + \frac{\sin\pi}{2} - \frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right)$$

$$\boxed{\text{Area } = \frac{\pi}{2} - 1}$$

b) Solve $\sin 2x \frac{dy}{dx} = y + \tan x$

Solution :

$$\frac{dy}{dx} - \frac{y}{sin2x} = \frac{tanx}{sin2x}$$

$\frac{dy}{dx} - \frac{y}{sin2x} = \frac{1}{2cos^2x}$

Comparing with $\frac{dy}{dx} + P(x)y = f(x)$

P(x) = $-\frac{1}{sin2x}$

f(x) = $\frac{1}{2cos^2x}$

I.F = $e^{\int \frac{-1}{sin} dx}$

= $e^{-\int cosec2xdx}$

= $e^{\frac{-\log(cosec2x - cot \quad)}{2}}$

= $e^{\frac{-\log\left(\frac{1-cos}{sin2}\right)}{2}}$

= $e^{\frac{-\log\left(\frac{2sin^2x}{2sinx.cosx}\right)}{2}}$

= $e^{\frac{-\log(tanx)}{2}}$

I.F. = $\frac{1}{\sqrt{tanx}}$

The solution is,

y x I.F. = $\int P(x).I.F.dx + c$

$\frac{y}{\sqrt{tanx}} = \int \frac{1}{2cos^2x}$ x $\frac{1}{\sqrt{tanx}} dx + c$

$\frac{y}{\sqrt{tanx}} = \int \frac{1}{2co \ ^2x}$ x $\frac{1}{\sqrt{tanx}} dx + c$

$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int \frac{1}{\sqrt{cos^4x.\frac{sinx}{cosx}}} + c$

$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int cos^{-3/2}x.sin^{-1/2}xdx + c$

Put $cos^{-\frac{1}{2}}x = t$

$\frac{1}{2}cos^{-3/2}x.sinxdx = dt$

$$\frac{1}{2}cos^{-3/2}x.sin^{-1/2}x.sin^{3/2}xdx = dt$$

$$\frac{1}{2}cos^{-3/2}x.sin^{-1/2}xdx = \frac{dt}{sin^{3/2}x} \qquad ......(1)$$

Now,

$$cos^{-\frac{1}{2}}x = t$$

$$t^{-4} = cos^2x$$

$$(1 - t^{-4}) = 1 - cos^2x$$

$$(1 - t^{-4}) = sin^2x$$

$$sin^{3/2}x = (1 - t^{-4})^{3/4} \qquad .....(2)$$

Substituting (2) in (1),

$$\frac{1}{2}cos^{-3/2}x.sin^{-1/2}xdx = \frac{dt}{(1-t^{-4})^{3/4}}$$

$$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int \frac{dt}{(1-t^{-4})^{\frac{3}{4}}} + c$$

$$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int \frac{t^3dt}{(t^4-1)^{3/4}} + c$$

Let $t^4 - 1 = g$

$$4t^3dt = dg$$

$$t^3dt = \frac{dg}{4}$$

$$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int \frac{dg}{4g^{3/4}} + c$$

$$\frac{y}{\sqrt{tanx}} = \frac{1}{8}\int g^{-3/4}dg + c$$

$$\frac{y}{\sqrt{tanx}} = \frac{1}{8}\frac{g^{1/3}}{1/3} + c$$

$$\frac{y}{\sqrt{tanx}} = \frac{3}{8}g^{1/3} + c$$

Substituting $g = t^4 - 1$

$$\frac{y}{\sqrt{tanx}} = \frac{3}{8}(t^4 - 1)^{1/3} + c$$

Substituing $t = cos^{-1/2}x$

$$\frac{y}{\sqrt{tanx}} = \frac{3}{8}[(cos^{-\frac{1}{2}}x)^4 - 1]^{1/3} + c$$

$$\boxed{\frac{y}{\sqrt{tanx}} = \frac{3}{8}[cos^{-2}x - 1]^{1/3} + c}$$

c) Solve $\frac{dy}{dx} = 3x + y^2$ with initial conditions y_0 = 1, x_0 = 0 at x = 0.2 in steps of h = 0.1 by Runge Kutta method of fourth order.

Solution :

$$\frac{dy}{dx} = 3x + y^2$$

f(x, y) = $3x + y^2, x_0 = 0, y_0 = 1, h = 0.1$

$k_1 = hf(x_0, y_0) = 0.1[3(0) + 1^2] = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right] = 0.1252$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1252}{2}\right)^2\right] = 0.1279$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1[3(0 + 0.1) + (1 + 0.1279)^2] = 0.1572$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \frac{1}{6}[0.1 + 2(0.1252) + 2(0.1279) + 0.1572]$$

$$k = \frac{1.2634}{6} = 0.2105$$

The approximate value of y at x = 0.2 is = $y_0 + k = 1 + 0.2105 = 1.2105$

Q.5

a) Evaluate $\int_0^1 x^5 \sin^{-1} x\, dx$ and find the value of $\beta\left(\frac{7}{2}, \frac{1}{2}\right)$.

Solution :

Integrating by parts we have,

$$\int_0^1 x^5 \sin^{-1} x\, dx = \left[\sin^{-1}x.\frac{x^6}{6}\right]_0^1 - \int_0^1 \frac{x^6}{6}.\frac{1}{\sqrt{1-x^2}}dx$$

$$\int_0^1 x^5 \sin^{-1} x\, dx = \frac{\pi}{2}.\frac{1}{6} - \frac{1}{6}\int_0^1 \frac{x^6}{\sqrt{1-x^2}}dx$$

Put x = sinθ dx = cosθdθ

$$I = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2}\frac{sin^6\theta}{cos\theta}cos\theta d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} sin^6\theta d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$= \frac{\pi}{12} - \frac{5\pi}{192}$$

$$I = \frac{11\pi}{192}$$

$$\boxed{\int_0^1 x^5 \sin^{-1} x\, dx = \frac{11\pi}{192}}$$

$$\boxed{\beta\left(\frac{7}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = \frac{\frac{7 X 5 X 3}{2 X 2 X 2}\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{3!} = \frac{35}{16}\Gamma^2\left(\frac{1}{2}\right)}$$

b) The differential equation of a moving body opposed by a force per unit mass of value cx and resistance per unit mass of value bv² where x and v are the displacement and velocity of the particle at that time is given by $v\frac{dv}{dx} = -cx - bv^2$. Find the velocity of the particle in terms of x, if it starts from rest.

Solution :

We have $v\frac{dv}{dx} = -cx - bv^2$

Putting $v^2 = y, v\frac{dv}{dx} = \frac{1}{2}\frac{dy}{dx}$

$\frac{1}{2}\frac{dy}{dx} + by = -cx$

$\frac{dy}{dx} + 2by = -2cx$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

I.F. = $e^{\int Pdx} = e^{\int 2bdx} = e^{2b}$

The solution is $ye^{2bx} = \int e^{2bx}(-2cx)dx + c'$

$ye^{2bx} = -2c\int xe^{2bx}dx + c'$

$ye^{2bx} = -2c\left(x\frac{e^{2bx}}{2b} - \int 1.\frac{e^{2bx}}{2b}dx\right) + c'$

$ye^{2bx} = -2c\left(x\frac{e^{2bx}}{2b} - \frac{e^{2bx}}{4b^2}\right) + c'$

Resubstituting $y = v^2$

$v^2e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2b} + c'$

By data, when x = 0, v = 0 So, $c' = -\frac{c}{2b^2}$

$v^2e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} - \frac{c}{2b^2}$

$\boxed{v^2 = \frac{c}{2b^2}(e^{2bx} - 1) - \frac{cx}{b}}$

c) Evaluate $\int_0^6 \frac{dx}{1+4x}$ by using i) Trapezoidal ii) Simpsons (1/3)rd and iii) Simpsons (3/8)th rule.

Solution :

Dividing the interval to 6 parts by taking each subinterval equal to $h = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
$y = \frac{1}{1+4x}$	1	$\frac{1}{5}$	$\frac{1}{9}$	$\frac{1}{13}$	$\frac{1}{17}$	$\frac{1}{21}$	$\frac{1}{25}$
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By Trapezoildal Rule,

$I = \frac{h}{2}[X + 2R]$

Now, X = sum of the extremes = $1 + \frac{1}{25} = 1.04$

And, R = sum of the remaining = $\frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \frac{1}{21} = 0.4944$

$$\boxed{I = \frac{h}{2}[X + 2R] = \frac{1}{2}[1.04 + 0.4944] = 0.7672}$$

ii) By Simpsons (1/3)rd rule,

$I = \frac{h}{3}[X + 2E + 4O]$

Now, X = sum of the extremes = $1 + \frac{1}{25} = 1.04$

2E = 2 x sum of the even ordinates = $2\left(\frac{1}{9} + \frac{1}{17}\right) = 0.3398$

4O = 4 x sum of the odd ordinates = $4\left(\frac{1}{5} + \frac{1}{13} + \frac{1}{21}\right) = 1.2981$

$$\boxed{I = \frac{h}{3}[X + 2E + 4O] = \frac{1}{3}[1.04 + 0.3398 + 1.2981] = 0.8926}$$

iii) By Simpsons (3/8)th rule,

$I = \frac{3h}{8}[X + 2T + 3R]$

Now, X = sum of the extremes = $1 + \frac{1}{25} = 1.04$

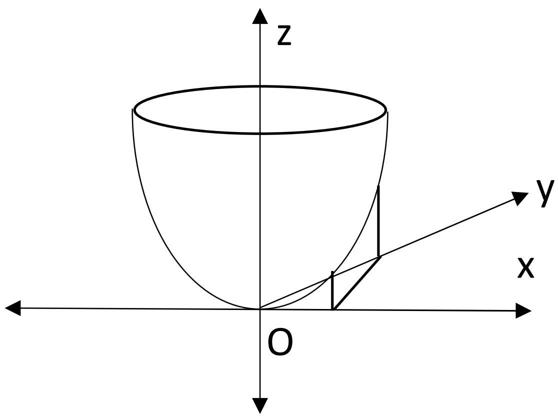
2T = 2 x sum of the multiples of 3 = $2 \times \frac{1}{13} = 0.1538$

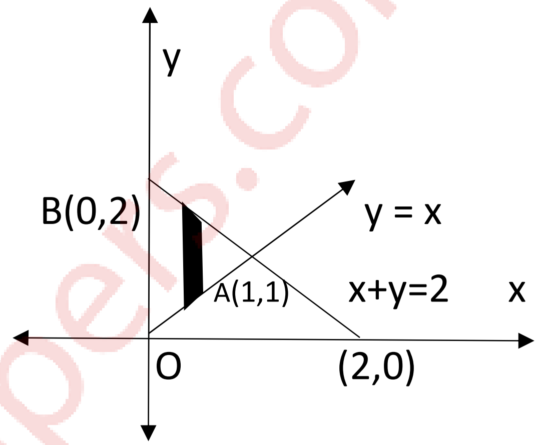
3R = 3 x sum of the remaining = $3\left(\frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{21}\right) = 1.2526$

$$\boxed{I = \frac{3h}{8}[X + 2T + 3R] = \frac{3}{8}[1.04 + 0.1538 + 1.2526] = 0.9174}$$

Q.6
a) Find the volume of the region that lies under the paraboloid z = x^2 + y^2 and above the triangle enclosed by the lines y = x, x = 0 and x + y = 2 in the xy plane.

Solution :





The base of the required solid is a triangle OAB.
Take a strip parallel to the y-axis from y = x to y = 2-x. The strip moves parallel to itself from x = 0 to x = 1. Z varies from 0 to x^2 + y^2.

V = $\int_0^1 \int_x^{2-x} \int_0^{x^2+y^2} dzdydx = \int_0^1 \int_x^{2-x} (x^2+y^2)dydx \;= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx$

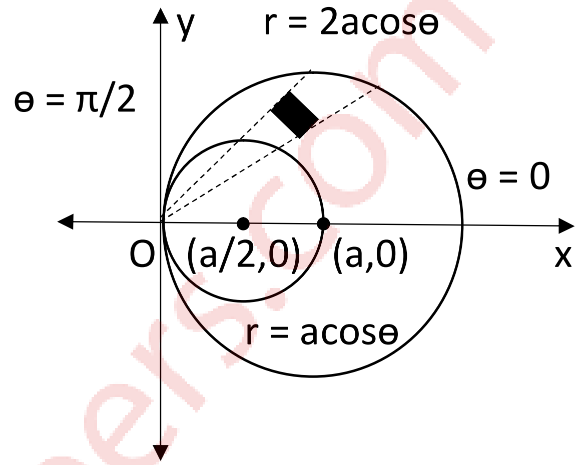
$= \int_0^1 \left[x^2(2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\right] dx = \int_0^1 2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}$dx

$= \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1 = \frac{2}{3} - \frac{7}{12} - \frac{1}{12} + \frac{16}{12} = \frac{4}{3}$

$\boxed{V = \frac{4}{3}}$

b) Change to polar coordinates and evaluate $\iint y^2 dxdy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$

Solution :



$$x^2 + y^2 - ax = 0$$

$$x^2 - ax + \left(\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

Centre ≡ (a/2,0)

Radius = a/2

And,

$$x^2 + y^2 - 2ax = 0$$

$$x^2 - 2ax + a^2 + y^2 = a^2$$

$$(x - a)^2 + y^2 = a^2$$

Centre ≡ (a,0)

Radius = a

Putting x=rcosɵ and y=rsinɵ in $x^2 + y^2 - ax = 0$ we get $r^2 = arcos\theta$ i.e. $r = acos\theta$ and in $x^2 + y^2 - 2ax = 0$ we get $r^2 = 2arcos\theta$ i.e. $r = 2acos\theta$

Considering a radial strip, r varies from acosɵ to 2acosɵ and ɵ varies from 0 to $\frac{\pi}{2}$.

I = $\iint y^2 dxdy$

I = $2\int_0^{\frac{\pi}{2}}\int_{\text{acosɵ}}^{\text{2acosɵ}} (rsin\theta)^2 rdrd\theta$

I = $2\int_0^{\frac{\pi}{2}}\int_{\text{acosɵ}}^{\text{2acosɵ}} r^3 sin^2\theta drd\theta$

I = $2\int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_{\text{acosɵ}}^{\text{2acosɵ}} sin^2\theta d\theta$

I = $\frac{1}{2}\int_0^{\frac{\pi}{2}}(16a^4cos^4\theta - a^4cos^4\theta)\, sin^2\theta d\theta$

I = $\frac{15a^4}{2}\int_0^{\frac{\pi}{2}} cos^4\theta sin^2\theta d\theta$

I = $\frac{15a^4}{2}$ x $\frac{3.1.1}{6.4.2}$ x $\frac{\pi}{2}$

$\boxed{I = \frac{15\pi a^4}{64}}$

c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$$

Solution :

The auxiliary equation is $D^2 + 1 = 0$

$D = i, -i$

Complementary Function, C.F. = $c_1\cos x + c_2\sin x$

Here $y_1 = \cos x$, $y_2 = \sin x$ and $X = \frac{1}{1+\sin x}$

Let Particular Integral, P.I = $uy_1 + vy_2$

Now,

$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = cos^2x + sin^2x = 1$

$u = -\int \frac{y_2X}{W} dx = -\int \frac{sinx}{1} \text{x} \frac{1}{1+sinx} dx = -\int \frac{sinx}{1-sinx} \text{x} \frac{1-sin}{1+sinx} dx = -\int \frac{sinx-sin^2x}{cos^2x} dx$

$= -\int (secx.tanx - tan^2x)\, dx = -\int (secx.tanx - sec^2x + 1)dx$

$u = -secx + tanx - x$

$v = \int \frac{y_1X}{W} dx = \int \frac{cosx}{1} \text{x} \frac{1}{1+s} dx$ = log(1 + sinx)

The complete solution is,

y = C.F. + P.I.

$\boxed{y = c_1\cos x + c_2\sin x + \cos x(-secx + tanx - x) + \sin x.\log(1 + \sin x)}$

MUMBAI UNIVERSITY PAPER SOLUTIONS

SEM II APPLIED MATHS II CBCGS DEC 2019

Q.P. Code: 29701

Q1)a) Evaluate $\int_0^\infty x e^{-x^4} dx$ **.** **(3M)**

Ans : Putting $x^4 = t$

$$x = t^{\frac{1}{4}}$$

$$dx = \frac{1}{4} t^{\frac{-3}{4}} dt$$

$$I = \int_0^\infty t^{\frac{1}{4}} . e^{-t} . \frac{1}{4} t^{\frac{-3}{4}} dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t} . t^{\frac{-1}{2}} dt = \frac{1}{4}\sqrt{\pi}$$

Q1)b) Find the length of the arc of the curve $r = a\sin^2\left(\frac{\theta}{2}\right)$ **from Θ = 0 to any point P(Θ).** **(3M)**

Ans : The required arc is given by

$$L = \int_0^{P(\theta)} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$L = \int_0^{P(\theta)} \sqrt{r^2 + \left(X\frac{1}{2} a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta$$

$$L = \int_0^{P(\theta)} \sqrt{r^2 + \left(X\frac{1}{2} a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)}\, d\theta$$

$$= \int_0^{P(\theta)} \sqrt{r^2 + ar \quad os^2\frac{\theta}{2}}\, d\theta = \int_0^{P(\theta)} r\sqrt{1 + \frac{a}{r}\cos^2\frac{\theta}{2}}\, d\theta$$

$$= \frac{r}{2}\left[\frac{1}{\sqrt{1 + \frac{a}{r}\cos^2\frac{\theta}{2}}} 2ar\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right]_0^{P(\theta)}$$

Q1)c) Solve $(D^4-2D^2+1)y=0$. **(3M)**

Ans : The auxiliary equation is

$$D^4-2D^2+1=0$$
$$\therefore \left(D^2-1\right)^2=0$$
$$\therefore D^2-1=0$$
$$\therefore D^2=1,1$$
$$\therefore D=1,-1,1,-1$$

The roots are real and repeated .

Therefore, $y=\left(c_1+c_2x\right)e^x+\left(c_3+c_4x\right)e^{-x}$

Q1)d) Solve $(x-2e^y)dy+(y+x\sin x)dx=0$. **(3M)**

Ans : We have M = y + xsinx and N = x – 2e^y

$$\therefore \frac{\partial M}{\partial y}=1=\frac{\partial N}{\partial x}$$
$$\therefore \int Mdx=\int\left(y+x\sin x\right)dx=yx+x(-\cos x)-\int(-\cos x).1.dx$$
$$=xy-x\cos x+\sin x$$
$$\therefore \int -2e^y dy=-2e^y$$

The solution $xy-x\cos x+\sin x-2e^y=c$.

Q1)e) Evaluate $\int_0^1\int_0^x x^2y^2(x+y)dydx$. **(4M)**

Ans :

$$\int_0^1\int_0^x (x^3y^2+x^2y^3)dydx$$
$$=\int_0^1\left[\frac{x^3y^3}{3}+\frac{x^2y^4}{4}\right]_0^x dx$$
$$=\int_0^1\left(\frac{x^6}{3}+\frac{x^6}{4}\right)dx=\frac{7}{12}\int_0^1 x^6dx=\left[\frac{7}{12}X\frac{x^7}{7}\right]_0^1=\frac{1}{12}$$

Q1)f) Solve $\frac{dy}{dx} = x^3 + y$ with initial conditions $x_0 = 1$, $y_0 = 1$ by Taylor's method. Find the approximate value of y for x=0.1. **(4M)**

Ans : The Taylor's Series is given by

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + ...$$

Now,

$$y' = x^3 + y$$
$$y_0' = x_0^3 + y_0 = 1^3 + 1 = 1$$
$$y'' = 3x^2 + y'$$
$$y_0'' = 3x_0^2 + y_0' = 3 + 1 = 4$$
$$y''' = 6x + y''$$
$$y_0''' = 6x_0 + y_0'' = 6 + 4 = 10$$

Putting these values in the series , we get

y= 1 + 0.1 X 1 + (0.01/2) X 4 + (0.001/6) X 10 + ……….

y =1.12167

The approximate value of y is 1.12167 .

Q2)a) Solve $\frac{d^2y}{dx^2} - 4y = x^2e^{3x} + e^{3x} - \sin 2x$. **(6M)**

Ans : The auxiliary equation is D^2-4=0 . Hence, D = 2, -2 .

The Complementary Equation is $y = c_1e^{2x} + c_2e^{-2x}$.

$$P.I. = \frac{1}{D^2-4}\left(x^2e^{3x} + e^{3x} - \sin 2x\right) .$$

$$\frac{1}{D^2-4}e^{3x}x^2 = e^{3x}.\frac{1}{D^2-4}.x^2 = e^{3x}.\frac{1}{(D+3)^2-4}.x^2 = e^{3x}.\frac{1}{D^2+6D+5}.x^2 .$$

$$= \frac{e^{3x}}{5}\left[1 + \frac{D^2+6D}{5}\right]^{-1}x^2 = \frac{e^{3x}}{5}\left[1 - \frac{D^2+6D}{5} + \frac{36D^2}{25} + ...\right]x^2$$

$$= \frac{e^{3x}}{5}\left[x^2 - \frac{12x}{5} - \frac{2}{5} + \frac{72}{25}\right] = \frac{e^{3x}}{5}\left[x^2 - \frac{12x}{5} + \frac{62}{25}\right]$$

$$\frac{1}{D^2-4}e^x = \frac{-1}{3}e^x$$
$$\frac{1}{D^2-4}\sin 2x = -\frac{1}{2}\sin 2x$$

The complete solution is

$$y = c_1e^{2x} + c_2e^{-2x} + \frac{e^{3x}}{5}\left[x^2 - \frac{12x}{5} + \frac{62}{25}\right] + \frac{-1}{3}e^x - \frac{1}{2}\sin 2x.$$

Q2)b) Show that $\int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a}$ **, (a>0)** **(6M)**

Ans : Let I(a) be the given integral. Then by the rule of differentiation under integration sign,

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}dx = \int_0^\infty \frac{1}{x^2}.\frac{1}{1+ax^2}.x^2dx = \int_0^\infty \frac{dx}{1+ax^2}$$
$$= \frac{1}{a}\int_0^\infty \frac{dx}{(1/a)+x^2} = \frac{1}{a}.(\sqrt{a})\left[\tan^{-1} x\sqrt{a}\right]_0^\infty = \frac{1}{\sqrt{a}}.\frac{\pi}{2}$$
$$\therefore \frac{dI}{da} = \frac{\pi}{2\sqrt{a}}$$

Integrating both sides , $I(a) = \frac{\pi}{2}\int \frac{da}{\sqrt{a}} = \pi\sqrt{a} + c$.

To find c , put a= 0. Hence, I(0)=c .

$$I(0) = \int_0^\infty 0dx = 0, \therefore c = 0, \therefore I = \pi\sqrt{a}$$

But

$$\therefore \int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a}.$$

Q2)c) Change the order of integration and evaluate $\int_0^5 \int_{2-x}^{x+2} dydx$ **.** **(8M)**

Ans : 1.Given order and given limits : Given order is first w.r.t y and then w.r.t x ,i.e. a strip parallel to the y-axis. Y changes from y=2-x to y=2+x and then x changes from x=0 to x=5.

2. Region of integration : y=2-x is a straight line x+y=2 and y=2+x is also a straight line. x=0 is the y-axis and x=5 is a line parallel to the y-axis .The points of intersection are A(0,2), B(5,-3) and C(5,7)

.The region of integration is the triangle ABC .

3. Change the order of integration : To change the order of integration, consider a strip parallel to the x-axis in the region of integration .When the strip moves parallel to itself, its base moves on two different straight lines AB and AC .Thus, the region of integration is split into two parts, ADC and ADB .So we consider two strips in the two regions. In the region ABD on the strip x varies from x= 2-x to x=5 and then the strip moves from y =-3 to y=2 .In the region ADC, on the strip x varies from x= y-2 to x =5 and then the strip moves from y =2 to y =7.

$$\therefore I = \int_{-3}^{2}\int_{2-y}^{5} dxdy + \int_{2}^{7}\int_{y-2}^{5} dxdy$$

$$= \int_{-3}^{2}[x]_{2-y}^{5} dy + \int_{2}^{7}[x]_{y-2}^{5} dy$$

$$= \int_{-3}^{2}(3+y)dy + \int_{2}^{7}(7-y)dy \quad .$$

$$= \left[3y + \frac{y^2}{2}\right]_{-3}^{2} + \left[7y - \frac{y^2}{2}\right]_{7}^{2}$$

$$= \left(17 - \frac{9}{2}\right) + \left(37 - \frac{49}{2}\right) = 25$$

Q3)a) Evaluate $\iiint zdxdydz$ over the volume of tetrahedron bounded by the planes x=0,y=0,z=0 and $\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=1$. **(6M)**

Ans : We put x=3u , y=4v , z=5w , dx=3du , dy=4dv , dz=5dw .

$$\therefore I = 60\iiint 5wdudvdw \ .$$

As before the limits of integration change and we have

$$\therefore I = 3X\,4X\,5^2 \int_{u=0}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} wdwdvdu$$

$$= 300\int_{u=0}^{1}\int_{v=0}^{1-u}\left[\frac{w^2}{2}\right]_0^{1-u-v} dvdu = \frac{300}{2}\int_{u=0}^{1}\int_{v=0}^{1}(1-u-v)^2 dvdx$$

$$= 150\int_{u=0}^{1}\left[-\frac{(1-u-v)^3}{3}\right]_0^{1-u} du = -\frac{150}{3}\int_0^1\left[0-(1-u)^3\right]du$$

$$= 50\int_0^1 (1-u)^3 du = 50\left[-\frac{(1-u)^4}{4}\right]_0^1 = -\frac{50}{4}[0-1] = \frac{25}{2}$$

Q3)b) Find the mass of the lamina bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ if the density of the lamina at any point varies as the square of its distance from the origin. **(6M)**

Ans : The two curves intersect at A(a,a). The lamina is the area OBACO. On the curve OCA, $y = \sqrt{4x} = 2\sqrt{x}$ and on the curve OBA , $y = \frac{x^2}{4}$. The surface density is given by

$\rho = k(x^2 + y^2)$ .Taking the elementary strip parallel to the y-axis , on the strip y varies from $y = \frac{x^2}{4}$ to $y = \sqrt{4x} = 2\sqrt{x}$ and then x varies from x=0 to x=4 .

Therefore Mass of the lamina =

$$= k\int_0^4 \int_{x^2/4}^{\sqrt{4x}} (x^2 + y^2)dxdy = k\int_0^a \left[x^2 y + \frac{y^2}{3} \right]_{x^2/4}^{\sqrt{4x}} dx$$

$$= k\int_0^a \left(x^2.\sqrt{4x} + (4x).\sqrt{4x} - x^2.\frac{x^2}{4} - \frac{1}{3}.\frac{x^6}{4^3} \right) dx$$

$$= k\int_0^a \left(2x^{5/2} + \frac{4.2}{3}.x^{3/2} - \frac{x^4}{4} - \frac{x^6}{3X\,4^3} \right) dx$$

$$= k\left[2.\frac{x^{\frac{7}{2}}}{7/2} + \frac{8}{3}.\frac{x^{5/2}}{5/2} - \frac{1}{4}.\frac{x^5}{5} - \frac{1}{3X\,64}.\frac{x^7}{7} \right]_0^a$$

$$= k\left[\frac{2}{7}.256 + \frac{2}{15}.256 - \frac{256}{5} - \frac{256}{21} \right]$$

$$= \frac{6X\,256k}{35} = \frac{1536}{35}k$$

Q3)c) Solve $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = -x^4 \sin x$. **(8M)**

Ans : Putting z= log x and x =e^z , we get

$$[D(D-1) - 4D + 6)y = -e^{4z}.\sin e^z.$$
$$\therefore (D^2 - 5D + 6)y = -e^{4z}.\sin e^z$$

The Auxiliary Equation is

$$\therefore (D^2 - 5D + 6) = 0$$
$$\therefore (D-2)(D-3) = 0$$
$$\therefore D = 2,3$$

The Complementary Function is

$$y = c_1 e^{2z} + c_2 e^{3z}$$

$$P.I. = \frac{1}{D^2 - 5D + 6}(-e^{4z}\sin e^z)$$

$$= -e^{4z}.\frac{1}{(D+4)^2 - 5(D+4) + 6}\sin e^z$$

$$= -e^{4z}.\frac{1}{D^2 + 3D + 2}\sin e^z = -e^{4z}.\frac{1}{(D+2)(D+1)}\sin e^z$$

$$= -e^{4z}.\frac{1}{D+2}.e^{-z}.\int e^z \sin e^z dz$$

Put e^z=t

$$\therefore P.I = -e^{4z}.\frac{1}{D+2}.e^{-z}(-\cos e^z)$$

$$= e^{4z}.e^{-2z}.\int e^{2z}.e^{-z}\cos e^z dz \quad .$$

$$= e^{2z}\int e^z \cos e^z dz = e^{2z}.\sin e^z$$

Put $e^z = t$

The complete solution is

$$y = c_1 e^{2z} + c_2 e^{3z} + e^{2z}\sin e^z = c_1 x^2 + c_2 x^3 + x^2 \sin x \ .$$

Q4)a) Find by double integration the area between the curves $y^2 = 4x$ and 2x-3y+4 = 0 . (6M)

Ans : We first solve the two equations to find the points of intersection .

We get

$$y^2 = 2(3y-4)$$

$$y^2 - 6y + 8 = 0 \quad .$$

$$\therefore (y-4)(y-2) = 0, \therefore y = 2, 4$$

When y=2 , x=1 ;when y=4 , x=4 .Let the points of intersection be A(1,2) and B(4,4) .

Now, consider a strip parallel to the y -axis. On this strip, y varies from $y = \frac{2x+4}{3}$ to $y = 2\sqrt{x}$.

Then x varies from x=1 to x=4 .

$$\therefore A=\int_{1}^{4}\int_{(2x+4)/3}^{2\sqrt{x}} dydx=\int_{1}^{4}[y]_{(2x+4)/3}^{2\sqrt{x}}dx$$

$$=\int_{1}^{4}\left[2\sqrt{x}-\frac{(2x+4)}{3}\right]dx$$

$$=\left[2.\frac{2}{3}x^{3/2}-\frac{x^2+4x}{3}\right]_1^4$$

$$\therefore A=\left(\frac{32}{3}-\frac{32}{3}\right)-\left(\frac{4}{3}-\frac{5}{3}\right)=\frac{1}{3}$$

Q4)b) Solve $(1+\sin y)\dfrac{dx}{dy}=2y\cos y-x(\sec y+\tan y)$. **(6M)**

Ans : The given equation can be written as

$$\frac{dx}{dy}+\frac{(\sec y+\tan y)}{1+\sin y}.x=\frac{2y\cos y}{1+\sin y}$$

$$\therefore \frac{dx}{dy}+\sec y.\frac{(1+\sin y)}{(1+\sin y)}.x=\frac{2y\cos y}{1+\sin y}$$

$$\therefore \frac{dx}{dy}+\sec y.x=\frac{2y\cos y}{1+\sin y}.$$

$$\therefore e^{\int Pdy}=e^{\int \sec y}=e^{\log(\sec y+\tan y)}=\sec y+\tan y=\frac{1+\sin y}{\cos y}$$

Therefore, the solution is

$$x.\frac{(1+\sin y)}{\cos y}=\int \frac{2y\cos y}{(1+\sin y)}.\frac{(1+\sin y)}{\cos y}dy+c$$

$$=\int 2ydy+c=y^2+c.$$

The solution is $x.(1+\sin y)=y^2\cos y+c\cos y$

Q4)c) Solve $\frac{dy}{dx} = x^2 + y^2$ with initial conditions $y_0 = 1$, $x_0 = 1$ at x=0.2 in steps of h=0.1 by Runge Kutta method of fourth order . **(8M)**

Ans : We have $\frac{dy}{dx} = x^2 + y^2$

$$\therefore f(x,y) = x + y^2, x_0 = 0, y_0 = 1, h = 0.1$$
$$k_1 = hf(x_0, y_0) = 0.1(0+1) = 0.1$$
$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1\left[(0+0.05)+(1+0.05)^2\right] = 0.1125$$
$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1\left[(0.05)+(1+0.05762)^2\right] = 0.11686 \quad .$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1[(0+0.1]+(1+0.11686)^2] = 0.13474$$
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$k = \frac{1}{6}[0.1 + 2(0.11525) + 2(0.11686) + 0.13474] = 0.1165$$

The approximate value of y will be 1+0.1165 = 1.1165 .

Again to find at x = 0.2, we repeat the same .

$$\therefore f(x,y) = x + y^2, x = 0.2, x_0 = 0.1, y_0 = 1.1165, h = 0.1$$
$$k_1 = hf(x_0, y_0) = 0.1(0.1^2 + 1.1165^2) = 0.13466$$
$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1\left[(0.1+0.05)+(1.1165+0.06733)^2\right] = 0.15514$$
$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1\left[(0.1+0.05)+(1.1165+0.07757)^2\right] = 0.15758$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1[(0.1+0.1]+(1.1165+0.15758)^2] = 0.18233$$
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$k = \frac{1}{6}[0.13466 + 2(0.15514) + 2(0.15758) + 0.18233] = 0.1571$$

The approximate value of y = 1.1165 + 0.1571 = 1.2736 .

Q5)a) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \,.\, \int_0^1 \frac{dx}{\sqrt{1-x^4}}$. **(6M)**

Ans : Put $x^4 = t$

$$\therefore x = t^{1/4}$$

$\therefore dx = \frac{1}{4} t^{-3/4} dt$ When x =0, t= 0 ; when x =1 , t =1

$$\therefore I = \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}} . \frac{1}{4} . t^{-3/4} dt . \int_0^1 \frac{1}{\sqrt{1-t}} . \frac{1}{4} . t^{-3/4} dt$$

$$= \int_0^1 \frac{1}{4} . t^{-1/4} (1-t)^{-1/2} dt . \int_0^1 \frac{1}{4} . t^{-3/4} (1-t)^{-1/2} dt$$

$$= \frac{1}{16} B\left(\frac{-1}{4}+1, \frac{-1}{2}+1\right) . B\left(\frac{-3}{4}+1, \frac{-1}{2}+1\right)$$

$$= \frac{1}{16} B\left(\frac{3}{4}, \frac{1}{2}\right) . B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{16} . \frac{\neg(3/4)\neg(1/2)}{\neg(5/4)} . \frac{\neg(1/4)\neg(1/2)}{\neg(3/4)}$$

$$= \frac{1}{16} . \frac{\sqrt{\pi}}{1/4 \neg(1/4)} . \neg(1/4) . \neg(1/2)$$

$$= \frac{1}{4}\left(\sqrt{\pi}\right)^2 = \frac{\pi}{4}$$

Q5)b) The distance x descended by a parachute satisfies the differential equation $\left(\frac{dx}{dt}\right)^2 = k^2\left(1 - e^{-2gx/k^2}\right)$ **where k and g are constants. If x=0 when t=0, show that** $x = \frac{k^2}{g} \log\cosh\left(\frac{gt}{k}\right)$ **.** **(6M)**

Ans : We have

$$\frac{dx}{dt} = k\sqrt{1 - e^{-2gx/k^2}}$$

$$\therefore \frac{dx}{\sqrt{1 - e^{-2gx/k^2}}} = kdt$$

Let $\sqrt{1-e^{-2gx/k^2}}=u$.

$\therefore 1-e^{-2gx/k^2}=u^2$.

$$\therefore e^{-2gx/k^2}.\frac{g}{k^2}.dx=udu$$

$$\therefore (1-u^2)\frac{g}{k^2}dx=udu$$

$$\therefore dx=\frac{k^2}{g}.\frac{u}{1-u^2}du$$

Hence we get that

$$\frac{k^2}{g}.\frac{u}{1-u^2}.\frac{1}{u}du=kdt$$.

$$\therefore \frac{k}{g}.\frac{du}{1-u^2}=dt+c$$

By integration, $\frac{k}{g}.\frac{1}{2}\log\left(\frac{1+u}{1-u}\right)=t+c$.

But

$$\frac{1}{2}\log\left(\frac{1+u}{1-u}\right)=\tanh^{-1}u$$.

$$\therefore \frac{k}{g}\tanh^{-1}u=t+c$$

But by data when t=0, x=0 and hence u=0 .Therefore , c=0 .

$$\therefore \frac{k}{g}\tanh^{-1}u=t$$

$$\tanh^{-1}u=\frac{gt}{k}$$

$$u=\tanh\frac{gt}{k}$$.

$$\therefore u^2=\tanh^2\left(\frac{gt}{k}\right)$$

$$\therefore 1-e^{-2gx/k^2}=\tanh^2\left(\frac{gt}{k}\right)$$

$$e^{-2gx/k^2}=1-\tanh^2\left(\frac{gt}{k}\right)=\sec h^2\left(\frac{gt}{k}\right)$$

$$e^{2gx/k^2} = \cos h^2\left(\frac{gt}{k}\right)$$

$$\therefore \frac{2gx}{k^2} = 2\log\cosh\left(\frac{gt}{k}\right)$$

$$\therefore x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$$

Q5)c) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ **by using i) Trapezoidal ii) Simpsons (1/3)rd and iii) Simpsons (3/8)th rule . (8M)**

Ans : Firstly, we shall divide the interval (0,1) into 10 equal parts by taking h=0.1. We prepare the following table :

x :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y :	1	0.9901	0.9615	0.9174	0.8621	0.8	0.7353	0.6711	0.6098	0.5525	0.5

(i) By Trapezoidal Rule

$$I = \frac{h}{2}[X + 2R]$$

X = 1.5 , R =7.0998

$$I = \frac{0.1}{2}[1.5 + 2X\,7.0998]$$

$$\therefore I = 0.7849$$

(ii) By Simpson's (1/3)rd rule

$$S = \frac{h}{3}[X + 2E + 4O]$$

X = 1.5 , E = 3.1687 , O = 3.9311

$$S = \frac{0.1}{3}[1.5 + 2X\,3.1687 + 4X\,3.9311]$$

S = 0.7853

(iii) By Simpson's (3/8)th rule

$$S = \frac{3h}{8}[X + 2T + 3R]$$

X = 1.5 , T= 2.2052 , R= 4.8946

$$S = \frac{3X\,0.1}{8}[1.5 + 2X\,2.2052 + 3X\,4.8946]$$

S = 0.7834

Q6)a) Find the volume in the first octant bounded by the cylinder $x^2+y^2=2$ and the planes $z=x+y, y=x, z=0$ and x=0 . **(6M)**

Ans : If we take projections on the xy – plane ,the area is bounded by the circle $x^2+y^2=2$, the line y=x and the line x=0 i.e. the y-axis .

We change the co-ordinates to cylindrical polar by putting x=rcosΘ , y=rsinΘ , z=z .

Then the equation of the cylinder becomes $x^2+y^2=2$ i.e. $r=\sqrt{2}$.

The line y = x becomes , $r\sin\theta = r\cos\theta, \therefore \theta = \frac{\pi}{4}$.

The line x=0 becomes, $r\cos\theta = 0, \therefore \theta = \frac{\pi}{2}$.

Now if we consider a radial strip in the projection , r varies from r=0 to $r=\sqrt{2}$, Θ varies from $\theta=\frac{\pi}{4}$ to $\theta=\frac{\pi}{2}$. Then z varies from z = 0 to z = (x+y) = r(cosΘ+sinΘ) .

$$\therefore V = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}}\int_{z=0}^{r(\cos\theta+\sin\theta)} r\,dr\,d\theta\,dz = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r[z]_0^{r(\cos\theta+\sin\theta)}\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r^2(\cos\theta+\sin\theta)\,dr\,d\theta = \int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$$

$$= \frac{2\sqrt{2}}{3}\int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\,d\theta = \frac{2\sqrt{2}}{3}[\sin\theta-\cos\theta]_{\pi/4}^{\pi/2}$$

$$= \frac{2\sqrt{2}}{3}\left[(1-0)-\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\right] = \frac{2\sqrt{2}}{3}$$

Q6)b) Change the polar coordinates and evaluate $\iint_R \frac{dxdy}{(1+x^2+y^2)^2}$ over one loop of the lemniscates $(x^2+y^2)^2=x^2-y^2$. **(6M)**

Ans : If we put $x=2\cos\theta$ and $y=r\sin\theta$, $(x^2+y^2)^2=x^2-y^2$ becomes

$$r^4 = r^2(\cos^2\theta-\sin^2\theta)\, i.e.$$
$$r^2=\cos 2\theta$$
$$\therefore \frac{1}{(1+x^2+y^2)^2} = \frac{1}{(1+r^2)^2}$$

Now, the loop varies from 0 to $\sqrt{\cos 2\theta}$ and Θ varies from $-\pi/4$ to $\pi/4$.

$$\therefore I = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{rdrd\theta}{(1+r^2)^2}$$

$$= 2\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{rdrd\theta}{(1+r^2)^2}$$

$$= -\int_0^{\pi/4}\left[\frac{1}{1+\cos 2\theta}-1\right]d\theta = \int_0^{\pi/4}\left[1-\frac{1}{1+\cos 2\theta}\right]d\theta$$

$$= \int_0^{\pi/4}\left[1-\frac{\sec^2\theta}{2}\right]d\theta = \left[\theta-\frac{\tan\theta}{2}\right]_0^{\pi/4} = \frac{\pi}{4}-\frac{1}{2} = \frac{\pi-2}{4}$$

Q6)c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}\ .$$ **(8M)**

Ans : The Auxiliary equation is

$$D^2 - 1 = 0$$
$$\therefore (D-1)(D+1) = 0 \quad .$$
$$\therefore D = 1, -1$$

The Complementary Function is $y = c_1e^x + c_2e^{-x}$.

Hence, $y_1 = e^x, y_2 = e^{-x}$ and $X = \frac{2}{1+e^x}$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2e^0 = -2$$

Now, $\therefore u = -\int \frac{y_2X}{W}dx = -\int \frac{e^{-x}}{2}.\frac{2}{1+e^x}dx = \int \frac{e^{-x}}{1+e^x}dx$.

Put $e^{-x} = t$

$e^{-x}dx = -dt$

$$\therefore u = -\int \frac{y_2X}{W}dx = -\int \frac{e^{-x}}{2}.\frac{2}{1+e^x}dx = \int \frac{e^{-x}}{1+e^x}dx \quad .$$

$$e^{-x}dx = -dt$$

$$\therefore u = -\int \frac{dt}{1+(1/t)} = -\int \frac{t}{1+t}dt$$

$$= -\int \frac{(t+1)-1}{t+1}dt = -\int 1dt + \int \frac{dt}{1+t}$$

$$= -t + \log(1+t) = -e^{-x} + \log(1+e^{-x})$$

$$v = \int \frac{y_1 X}{W}dx = \int \frac{e^x}{-2}.\frac{2}{1+e^x}dx = -\int \frac{e^x}{1+e^x}dx = -\log(1+e^x)$$

$$\therefore P.I = uy_1 + vy_2 = \left[-e^{-x} + \log(1+e^{-x})\right]e^x + \left[-\log(1+e^x)\right]e^{-x}$$

The complete solution is

$$y = c_1e^x + c_2e^{-x} - 1 + e^x.\log(1+e^{-x}) - e^{-x}.\log(1+e^x) \quad .$$
